UNIVERSAL DEFORMATION RINGS AND DIHEDRAL DEFECT GROUPS

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Abstract. Let k be an algebraically closed field of characteristic 2, and let W be the ring of infinite Witt vectors over k. Suppose G is a finite group, and B is a block of kG with dihedral defect group D which is Morita equivalent to the principal 2-modular block of a finite simple group. We determine the universal deformation ring $R(G,V)$ for every kG-module V which belongs to B and has stable endomorphism ring k. It follows that $R(G,V)$ is always isomorphic to a subquotient ring of WD. Moreover, we obtain an infinite series of examples of universal deformation rings which are not complete intersections.

1. Introduction

In this paper we determine the universal deformation rings $R(G,V)$ associated to certain mod 2 representations V of finite groups G which belong to blocks of G having dihedral defect group D. There are three reasons for making this calculation. The first is that it provides more evidence for an affirmative answer to a question posed in [6], which would bound $R(G,V)$ in terms of the group ring of D over the Witt vectors (see Question 1.1). The second reason is that we produce an infinite family of $R(G,V)$ which are not complete intersections, but which do satisfy the dimension versus depth condition conjectured in [8] (see Question 1.3). This in turn leads to interesting questions in number theory (see the discussion after Question 1.3). The final reason is to describe a general method for applying results in modular and ordinary representation theory due to Brauer, Erdmann and others to the computation of universal deformation rings.

To make this more precise, let k be an algebraically closed field of characteristic $p > 0$, let $W = W(k)$ be the ring of infinite Witt vectors over k, and let F be the fraction field of W. Let Γ be a profinite group, and suppose V is a finite dimensional vector space over k with a continuous Γ-action. If all continuous $k\Gamma$-module endomorphisms of V are given by scalar multiplications, an argument of Faltings (see [18]) shows that V has a universal deformation ring $R(\Gamma,V)$. The topological ring $R(\Gamma,V)$ is universal with respect to deformations of V over commutative local W-algebras with residue field k which are the projective limits of their discrete Artinian quotients. For more information on deformation rings see [18], [27] and §2. In number theory, deformation rings are at the center of work by many authors concerning Galois representations, modular forms, elliptic curves and diophantine geometry (see e.g. [16], [34, 32], [10] and their references).

In [18], de Smit and Lenstra show that $R(\Gamma,V)$ is the inverse limit of the universal deformation rings $R(G,V)$ when G runs over all finite discrete quotients of Γ through which the Γ-action on V factors. Thus to answer questions about the ring structure of $R(\Gamma,V)$, it is natural to first consider the case when $\Gamma = G$ is finite. In this case, V has a universal deformation ring $R(G,V)$ under the weaker condition that the stable endomorphism ring $\underline{\mathrm{End}}_{kG}(V)$ is of dimension 1 over k (see [6, Prop. 2.1]); we assume $\underline{\mathrm{End}}_{kG}(V) = k$ in what follows.

In [6], the author and T. Chinburg determined $R(G,V)$ for V belonging to blocks with cyclic defect groups, i.e. blocks of finite representation type. In [4], the author considered V belonging to blocks with Klein four defect groups and described their universal deformation rings. This is a natural progression from [6], since the blocks with Klein four defect groups have tame representation type and are the only such blocks having abelian (non-cyclic) defect groups. The results obtained in [6] led to the following question.

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Question 1.1. *Let B be a block of kG with defect group D, and suppose V is a finitely generated kG-module with stable endomorphism ring k such that the unique (up to isomorphism) non-projective indecomposable summand of V belongs to B. Is the universal deformation ring $R(G,V)$ of V isomorphic to a subquotient ring of the group ring WD?*

It is shown in [6] and [4] that the answer to this question is positive in case D is cyclic or a Klein four group. Moreover, it follows that in all these cases $R(G,V)$ is a complete intersection ring (see [8, Thm. 7.2]). In case k has characteristic 2, G is the symmetric group S_4 and E is the unique (up to isomorphism) non-trivial simple kS_4-module, the author and T. Chinburg showed in [7, 8] that $R(S_4,E)\cong W[t]/(t^2,2t)$. Hence $R(S_4,E)$ is not a complete intersection ring, which answers a question of M. Flach [14]. Note that E belongs to the principal block of kS_4 which has as defect groups dihedral groups of order 8. A new proof of this result has been given by Byszewski in [13] using only elementary obstruction calculus. In [7, 8], the author and T. Chinburg also considered the question of whether deformation rings which are not complete intersections arise from arithmetic. It was shown in particular that there are infinitely many real quadratic fields L such that the Galois group $G_{L,\emptyset}$ of the maximal totally unramified extension of L surjects onto S_4 and $R(G_{L,\emptyset},E)\cong R(S_4,E)\cong W[t]/(t^2,2t)$ is not a complete intersection, where E is viewed as a module of $G_{L,\emptyset}$ via inflation.

In this paper, we expand these results further by considering entire families of blocks of tame representation type with defect $d\geq 3$. More precisely, our goal is to determine the structure of the universal deformation rings $R(G,V)$ in case D is dihedral of order at least 8 and B is Morita equivalent to the principal 2-modular block of a finite simple group. In particular, B contains precisely three isomorphism classes of simple modules. Note that in [9], Brauer has proved that a block with dihedral defect groups contains at most three simple modules up to isomorphism; hence we look at the largest case. It follows by the classifications by Gorenstein-Walter [22] and by Erdmann [20] that except for possibly one remaining family, all blocks B with dihedral defect groups containing precisely three isomorphism classes of simple modules are Morita equivalent to the principal 2-modular block of some finite simple group (see Remark 3.1).

A summary of our main results is as follows. The precise statements can be found in Propositions 4.1.1, 4.2.1 and 4.2.4 and in Theorem 5.1 and Corollary 5.1.2.

Theorem 1.2. *Suppose k has characteristic 2. Let B be a block of kG with dihedral defect group D of order 2^d where $d\geq 3$, which is Morita equivalent to the principal 2-modular block of a finite simple group. Let V be a finitely generated B-module with stable endomorphism ring k and universal deformation ring $R(G,V)$. Then either*

i. *$R(G,V)/2R(G,V)\cong k$, in which case $R(G,V)$ is isomorphic to a quotient ring of W, or*
ii. *$R(G,V)/2R(G,V)\cong k[t]/(t^{2^{d-2}})$, in which case $R(G,V)\cong W[[t]]/(p_d(t)(t-2),2\,p_d(t))$ for a certain monic polynomial $p_d(t)\in W[t]$ of degree $2^{d-2}-1$ whose non-leading coefficients are all divisible by 2.*

In all cases, $R(G,V)$ is isomorphic to a subquotient ring of WD. Given the block B, each of the cases (i) and (ii) occurs for infinitely many V. In case (ii), $R(G,V)$ is not a complete intersection.

This Theorem also gives an affirmative answer to the following question from [8] for B, D and V as in the statement of the Theorem (see [8, Thm. 7.2]).

Question 1.3. *Suppose B, D and V are as in Question 1.1 and D has nilpotency r. Is it the case that $\dim(R(G,V))-\mathrm{depth}(R(G,V))\leq r-1$?*

Theorem 1.2 provides an infinite series of finite groups G and mod 2 representations V for which $R(G,V)$ is not a complete intersection (see Corollary 5.1.2). This raises the question of whether one can use such G and V to construct further examples of deformation rings arising from arithmetic which are not complete intersections, in the following sense. As in [8], one can ask whether there are number fields L together with a finite set of places S of L such that G is a quotient of the Galois group $G_{L,S}$ of the maximal unramified outside S extension L_S of L which has the following property.

There should be a surjection $\psi : G_{L,S} \to G$ which induces an isomorphism $R(G_{L,S}, V) \to R(G, V)$ of deformation rings when V is viewed as a representation for $G_{L,S}$ via ψ. It was shown in [8] that a sufficient condition for $R(G_{L,S}, V) \to R(G, V)$ to be an isomorphism is that $\mathrm{Ker}(\psi)$ has no non-trivial pro-2 quotient. This is equivalent to the requirement that if L' is the fixed field of $\mathrm{Ker}(\psi)$ acting on L_S, then each ray class group of L' associated to a conductor involving only places over S should have odd order.

As mentioned earlier, the above arithmetic problem was considered in [8] when $G = S_4$ and V is irreducible of dimension 2. It is more challenging to treat the cases produced by Theorem 1.2 in an analogous way, but we think this raises interesting questions in Galois theory. For example, if $G = A_7$ and V is an irreducible mod 2 representation of degree 14, can one find L and S and $\psi : G_{L,S} \to G$ as above for which $\mathrm{Ker}(\psi)$ has no non-trivial pro-2 quotient, and hence $R(G_{L,S}, V) \cong R(G, V)$ is not a complete intersection? Is this possible when $L = \mathbb{Q}$? Another interesting case provided by Theorem 1.2 is when G is isomorphic to $\mathrm{PSL}_2(\mathbb{F}_q)$ where q is an odd prime power and 8 divides $\#G$. For example, if $q = \ell^2$ where $\ell \not\equiv \pm 1 \mod 24$ (resp. $\ell \not\equiv 1, 4, 16 \mod 21$), it was proved in [31] (resp. in [19]) that $\mathrm{PSL}_2(\mathbb{F}_q)$ occurs regularly over $\mathbb{Q}(t)$, implying that there are $\mathrm{PSL}_2(\mathbb{F}_q)$ extensions of any number field L for such q. On the other hand, it was shown in [33] that for any prime ℓ there are infinitely many positive integers r such that for $q = \ell^r$, $\mathrm{PSL}_2(\mathbb{F}_q)$ occurs as a Galois group over $\mathbb{Q}$. We are looking for $\mathrm{PSL}_2(\mathbb{F}_q)$ extensions of number fields which satisfy additional constraints on their ray class groups. Generalizing the techniques of [8] to treat such questions is beyond the scope of this paper, but we believe this will lead to interesting new number theoretic results.

We now describe the steps used to prove Theorem 1.2.

We first determine which indecomposable B-modules have stable endomorphism ring k, using that B is Morita equivalent to a special biserial algebra. This enables us to use the description of indecomposable modules of special biserial algebras as so-called string and band modules. We go through each component $\mathfrak{C}$ of the stable Auslander-Reiten quiver of B starting with modules in $\mathfrak{C}$ of minimal length. As it turns out, for each block B there are infinitely many isomorphism classes of indecomposable B-modules with stable endomorphism ring k. More precisely, there are always two components $\mathfrak{C}$ of type $\mathbb{Z}A_\infty^\infty$ which consist entirely of modules with stable endomorphism ring k. The other modules with stable endomorphism ring k either lie at the ends of 3-tubes, or they form a single Ω-orbit in one or two components $\mathfrak{C}'$ of type $\mathbb{Z}A_\infty^\infty$.

After we have found all indecomposable B-modules V with stable endomorphism ring k, we then determine their universal deformation rings modulo 2, i.e. $R(G, V)/2R(G, V)$. To do so, we concentrate first on one block B of a given defect $d \geq 3$. We then make use of the fact that all the blocks B of defect d are stably equivalent by a stable equivalence of Morita type over k. This leads to the universal deformation rings $R(G, V)$ in case (i) of Theorem 1.2.

For V as in case (ii), our strategy is to use Brauer's results on the ordinary characters belonging to B to find the largest quotient of $R(G, V)$ which is flat over W, namely $W[[t]]/(p_d(t))$ for $p_d(t)$ as in part (ii) of Theorem 1.2. We then use ring theory to show $R(G, V)$ must have the form $W[[t]]/(p_d(t)\,(t - 2\gamma), \alpha\, 2^m\, p_d(t))$ for some $\gamma \in W$, $\alpha \in \{0, 1\}$ and $m \geq 1$. To determine γ, α and m, we take advantage of the fact that if $\overline{U}$ is the universal mod 2 deformation of V then $\underline{\mathrm{End}}_{kG}(\overline{U}) = k$, so that $R(G, \overline{U})$ is well defined. To compute $R(G, \overline{U})$, we use that $\overline{U}$ lies at the end of a 3-tube of the stable Auslander-Reiten quiver of B. Using results from Brauer and Erdmann we see that the vertices of $\overline{U}$ are Klein four groups. Moreover, suppose K is one such Klein four group, and let $N_G(K)$ be the normalizer of K in G. Then the Green correspondent $f\overline{U}$ is a $kN_G(K)$-module which is induced from a module belonging to the end of a 3-tube of the stable Auslander-Reiten quiver of a block b_1 which is Morita equivalent to kA_4. The results of [4] imply that $R(G, \overline{U}) = k$, which leads to $\alpha = 1$ and $m = 1$.

The paper is organized as follows. In §2, we recall the definitions of deformations and deformation rings and prove that stable equivalences of Morita type preserve deformation rings (see Lemmas 2.2.2 and 2.2.3). We also prove some results which help determine universal deformation rings that are certain quotient rings of $W[[t]]$ (see Lemmas 2.3.1, 2.3.2, 2.3.3 and 2.3.6). In §3, we use the

classifications by Gorenstein-Walter [22] and by Erdmann [20] to describe all 2-modular blocks B of a finite group G which have dihedral defect groups and are Morita equivalent to the principal 2-modular block of a finite simple group. We also describe results by Brauer [9] about the ordinary irreducible characters of G belonging to B. In §4 and §6, we determine which indecomposable B-modules V have stable endomorphism ring k and find their universal deformation rings modulo 2 (see Propositions 4.1.1, 4.2.1 and 4.2.4). In §5, we analyze the B-modules belonging to the boundaries of 3-tubes and use the results from [9] about the ordinary irreducible characters belonging to B to determine the universal deformation rings of V (see Theorem 5.1). In particular, this implies Theorem 1.2. Since we make use of the fact that all the blocks we consider in this paper are Morita equivalent to special biserial algebras, we recall in §7 the basic definitions of special biserial algebras and string algebras and describe their indecomposable modules and their Auslander-Reiten quivers.

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2. Preliminaries: Universal deformation rings

In this section, we give a brief introduction to versal and universal deformation rings and deformations. For more background material, we refer the reader to [27] and [18].

Let k be an algebraically closed field of characteristic $p > 0$, and let W be the ring of infinite Witt vectors over k. Let $\hat{\mathcal{C}}$ be the category of all complete local commutative Noetherian rings with residue field k. The morphisms in $\hat{\mathcal{C}}$ are continuous W-algebra homomorphisms which induce the identity map on k. Let $\mathcal{C}$ be the full subcategory of $\hat{\mathcal{C}}$ of Artinian objects.

2.1. **Universal and versal deformation rings.** Suppose G is a finite group and V is a finitely generated kG-module. A lift of V over an object R in $\hat{\mathcal{C}}$ is a pair (M, ϕ) where M is a finitely generated RG-module which is free over R, and $\phi : k \otimes_R M \to V$ is an isomorphism of kG-modules. Two lifts (M, ϕ) and (M', ϕ') of V over R are isomorphic if there is an isomorphism $\alpha : M \to M'$ with $\phi = \phi' \circ (k \otimes \alpha)$. The isomorphism class $[M, \phi]$ of a lift (M, ϕ) of V over R is called a deformation of V over R, and the set of such deformations is denoted by $\mathrm{Def}_G(V, R)$. The deformation functor

$$\hat{F}_V : \hat{\mathcal{C}} \to \mathrm{Sets}$$

sends an object R in $\hat{\mathcal{C}}$ to $\mathrm{Def}_G(V, R)$ and a morphism $f : R \to R'$ in $\hat{\mathcal{C}}$ to the map $\mathrm{Def}_G(V, R) \to \mathrm{Def}_G(V, R')$ defined by $[M, \phi] \mapsto [R' \otimes_{R,f} M, \phi']$, where $\phi' = \phi$ after identifying $k \otimes_{R'} (R' \otimes_{R,f} M)$ with $k \otimes_R M$.

In case there exists an object $R(G, V)$ in $\hat{\mathcal{C}}$ and a deformation $[U(G, V), \phi_U]$ of V over $R(G, V)$ such that for each R in $\hat{\mathcal{C}}$ and for each lift (M, ϕ) of V over R there is a unique morphism $\alpha : R(G, V) \to R$ in $\hat{\mathcal{C}}$ such that $\hat{F}_V(\alpha)([U(G, V), \phi_U]) = [M, \phi]$, then $R(G, V)$ is called the universal deformation ring of V and $[U(G, V), \phi_U]$ is called the universal deformation of V. In other words, $R(G, V)$ represents the functor $\hat{F}_V$ in the sense that $\hat{F}_V$ is naturally isomorphic to $\mathrm{Hom}_{\hat{\mathcal{C}}}(R(G, V), -)$. In case the morphism $\alpha : R(G, V) \to R$ relative to the lift (M, ϕ) of V over R is not unique, $R(G, V)$ is called the versal deformation ring of V and $[U(G, V), \phi_U]$ is called the versal deformation of V.

By [27], every finitely generated kG-module V has a versal deformation ring. By a result of Faltings (see [18, Prop. 7.1]), V has a universal deformation ring in case $\mathrm{End}_{kG}(V) = k$.

The following two results were proved in [6], where Ω denotes the Heller operator for kG (see for example [1, §20]).

Proposition 2.1.1. ([6, Prop. 2.1]). *Suppose V is a finitely generated kG-module with stable endomorphism ring $\underline{\mathrm{End}}_{kG}(V) = k$. Then V has a universal deformation ring $R(G,V)$.*

Lemma 2.1.2. ([6, Cors. 2.5 and 2.8]). *Let V be a finitely generated kG-module with stable endomorphism ring $\underline{\mathrm{End}}_{kG}(V) = k$.*

i. *Then $\underline{\mathrm{End}}_{kG}(\Omega(V)) = k$, and $R(G,V)$ and $R(G,\Omega(V))$ are isomorphic.*
ii. *There is a non-projective indecomposable kG-module V_0 (unique up to isomorphism) such that $\underline{\mathrm{End}}_{kG}(V_0) = k$, V is isomorphic to $V_0 \oplus P$ for some projective kG-module P, and $R(G,V)$ and $R(G,V_0)$ are isomorphic.*

The following result can be proved similarly to [6, Prop. 5.2], using [6, Lemmas 5.3 and 5.4].

Proposition 2.1.3. *Let L be a subgroup of G and let U be a finitely generated indecomposable kL-module with $\underline{\mathrm{End}}_{kL}(U) = k$. Suppose there exists an indecomposable kG-module V with $\underline{\mathrm{End}}_{kG}(V) = k$ and a projective kG-module P such that*

$$\mathrm{Ind}_L^G U = V \oplus P. \tag{2.1.1}$$

Assume further that

$$\dim_k \mathrm{Ext}^1_{kL}(U,U) = \dim_k \mathrm{Ext}^1_{kG}(V,V). \tag{2.1.2}$$

Then $R(G,V)$ is isomorphic to $R(L,U)$.

2.2. **Stable equivalences of Morita type.** Suppose G (resp. H) is a finite group, and let A (resp. B) be a block of WG (resp. WH). For $R \in \mathrm{Ob}(\hat{\mathcal{C}})$, define RA (resp. RB) to be the block algebra in RG (resp. RH) corresponding to A (resp. B), i.e. $RA = R \otimes_W A$ (resp. $RB = R \otimes_W B$). Let Γ be A or B. Then Γ is a W-algebra that is projective as a W-module. Moreover, Γ is a symmetric W-algebra in the sense that Γ is isomorphic to its W-linear dual $\check{\Gamma} = \mathrm{Hom}_W(\Gamma, W)$, as Γ-Γ-bimodules. In the following, Γ-mod denotes the category of finitely generated left Γ-modules, and Γ-$\underline{\mathrm{mod}}$ denotes the W-stable category, i.e. the quotient category of Γ-mod by the subcategory of relatively W-projective modules. Recall that a Γ-module is called relatively W-projective if it is isomorphic to a direct summand of $\Gamma \otimes_W M$ for some W-module M.

In [5], it was shown that a split-endomorphism two-sided tilting complex (as introduced by Rickard [29]) for the derived categories of bounded complexes of finitely generated modules over A, resp. B, preserves the versal deformation rings of bounded complexes of finitely generated modules over kA, resp. kB. It follows from a result by Rickard (see [28] and [24, Prop. 6.3.8]) that a derived equivalence between the derived categories of bounded complexes $D^b(A\text{-mod})$ and $D^b(B\text{-mod})$ induces a stable equivalence between A-$\underline{\mathrm{mod}}$ and B-$\underline{\mathrm{mod}}$ which is itself induced by a bimodule. More precisely we get the following definition of stable equivalence of Morita type going back to Broué [11].

Definition 2.2.1. Let M be a B-A-bimodule and N an A-B-bimodule. We say M and N induce a stable equivalence of Morita type between A and B, if M and N are projective both as left and as right modules, and if

$$\begin{aligned} N \otimes_B M &\cong A \oplus P \quad \text{as } A\text{-}A\text{-bimodules, and} \\ M \otimes_A N &\cong B \oplus Q \quad \text{as } B\text{-}B\text{-bimodules,} \end{aligned} \tag{2.2.1}$$

where P is a projective A-A-bimodule, and Q is a projective B-B-bimodule. In particular, $M \otimes_A -$ and $N \otimes_B -$ induce mutually inverse equivalences between the W-stable module categories A-$\underline{\mathrm{mod}}$ and B-$\underline{\mathrm{mod}}$.

We now prove that stable equivalences of Morita type preserve versal deformation rings.

Lemma 2.2.2. *Let A and B be blocks of group rings over W as above. Suppose that M is a B-A-bimodule and N is an A-B-bimodule which induce a stable equivalence of Morita type between A and B. Let V be a finitely generated kA-module, and define $V' = (k \otimes_W M) \otimes_{kA} V$. Then $R(G,V)$ is isomorphic to $R(H,V')$ in $\hat{\mathcal{C}}$.*

Proof. Suppose $R \in \mathrm{Ob}(\mathcal{C})$ is Artinian. Then $M_R = R \otimes_W M$ is projective as left RB-module and as right RA-module, and $N_R = R \otimes_W N$ is projective as left RA-module and as right RB-module. Since $M_R \otimes_{RA} (N_R) \cong R \otimes_W (M \otimes_A N)$, we have, using (2.2.1),

$$M_R \otimes_{RA} N_R \cong RB \oplus P_R \quad \text{ as } RA\text{-}RA\text{-bimodules,}$$

where $P_R = R \otimes_W P$ is a projective RA-RA-bimodule. Similarly, we get

$$N_R \otimes_{RB} M_R \cong RA \oplus Q_R \quad \text{ as } RB\text{-}RB\text{-bimodules,}$$

where $Q_R = R \otimes_W Q$ is a projective RB-RB-bimodule.

Let $X = Q_k \otimes_{kA} V$ and $Y = P_k \otimes_{kB} V'$. Then since Q_k is a projective kA-kA-bimodule, it follows that X is a projective kA-module. Similarly, Y is a projective kB-module. Let X_R be the projective RA-cover of X and Y_R the projective RB-cover of Y, which exist since we assume R to be Artinian. Then X_R defines a lift (X_R, π_X) of X over R and Y_R defines a lift (Y_R, π_Y) over R. Because R is commutative local, every lift of X (resp. Y) over R is isomorphic to (X_R, π_X) (resp. (Y_R, π_Y)).

Let now (U, ϕ) be a lift of V over R. Then U is a finitely generated RA-module. Define $U' = M_R \otimes_{RA} U$. Since M is a finitely generated projective right RA-module and since U is a finitely generated free R-module, it follows that U' is a finitely generated projective, and hence free, R-module. Moreover

$$U' \otimes_R k = (M_R \otimes_{RA} U) \otimes_R k = M_k \otimes_{kA} (U \otimes_R k) \xrightarrow{M_k \otimes \phi} M_k \otimes_{kA} V = V'. \tag{2.2.2}$$

This means that $(U', \phi') = (M_R \otimes_{RA} U, M_k \otimes \phi)$ is a lift of V' over R. We therefore obtain for all $R \in \mathrm{Ob}(\mathcal{C})$ a well-defined map

$$\tau_R : \mathrm{Def}_G(V, R) \to \mathrm{Def}_H(V', R).$$

We need to show that τ_R is bijective. Let $U'' = N_R \otimes_{RB} U'$ and $\phi'' = N_k \otimes \phi'$. Similarly to (2.2.2), it follows that $(U'', \phi'') = (N_R \otimes_{RB} U', N_k \otimes \phi')$ is a lift of $V'' = N_k \otimes_{kB} V'$ over R. We have

$$\begin{aligned} (U'', \phi'') &= (N_R \otimes_{RB} (M_R \otimes_{RA} U), N_k \otimes (M_k \otimes \phi)) \\ &\cong ((RA \oplus Q_R) \otimes_{RA} U, (kA \oplus Q_k) \otimes \phi) \\ &\cong (U \oplus (Q_R \otimes_{RA} U), \phi \oplus (Q_k \otimes \phi)) \\ &\cong (U \oplus X_R, \phi \oplus \pi_X) \end{aligned} \tag{2.2.3}$$

as lifts of V'' over R, where the last isomorphism follows, since $(Q_R \otimes_{RA} U, Q_k \otimes \phi)$ is a lift of $X = Q_k \otimes_{kA} V$ over R. Moreover,

$$V'' = N_k \otimes_{kB} (M_k \otimes_{kA} V) \cong (kA \oplus Q_k) \otimes_{kA} V \cong V \oplus (Q_k \otimes_{kA} V) = V \oplus X.$$

Hence it follows by [6, Prop. 2.6] that τ_R is injective.

Now let (L, ψ) be a lift of $V' = M_k \otimes_{kA} V$ over R. Then $(L', \psi') = (N_R \otimes_{RB} L, N_k \otimes \psi)$ is a lift of $V'' = N_k \otimes_{kB} V' \cong V \oplus X$ over R. By [6, Prop. 2.6], there exists a lift (U, ϕ) of V over R such that $(L', \psi') \cong (U \oplus X_R, \phi \oplus \pi_X)$. Arguing similarly as in (2.2.3), we then have that (L', ψ') is isomorphic to $(U'', \phi'') = (N_R \otimes_{RB} U', N_k \otimes \phi')$ where $(U', \phi') = (M_R \otimes_{RA} U, M_k \otimes \phi)$. Therefore, $(M_R \otimes_{RA} L', M_k \otimes \psi') \cong (M_R \otimes_{RA} U'', M_k \otimes \phi'')$. Arguing again similarly as in (2.2.3), we have

$$\begin{aligned} (M_R \otimes_{RA} L', M_k \otimes \psi') &\cong (L \oplus Y_R, \psi \oplus \pi_Y), \text{ and} \\ (M_R \otimes_{RA} U'', M_k \otimes \phi'') &\cong (U' \oplus Y_R, \phi' \oplus \pi_Y). \end{aligned}$$

Thus by [6, Prop. 2.6], it follows that $(L, \psi) \cong (U', \phi')$, i.e. τ_R is surjective. Since the deformation functors $\hat{F}_V$ and $\hat{F}_{V'}$ are continuous, this implies that they are naturally isomorphic. Hence the versal deformation rings $R(G, V)$ and $R(H, V')$ are isomorphic in $\hat{\mathcal{C}}$. □

Using the same type of arguments as in the proof of Lemma 2.2.2, but restricting our attention to Artinian objects R in $\mathcal{C}$ that are k-algebras, we get the following weaker result about versal deformation rings modulo p.

Lemma 2.2.3. *Let A and B be blocks of group rings over W as above. Suppose that M_k is a kB-kA-bimodule and N_k is a kA-kB-bimodule which induce a stable equivalence of Morita type between kA and kB. Let V be a finitely generated kA-module, and define $V' = M_k \otimes_{kA} V$. Then $R(G,V)/pR(G,V) \cong R(H,V')/pR(H,V')$.*

Remark 2.2.4. Using the notations of Lemma 2.2.2 (resp. of Lemma 2.2.3), suppose that the stable endomorphism ring $\underline{\mathrm{End}}_{kG}(V) = k$. Then it follows that also $\underline{\mathrm{End}}_{kH}(V') = k$. By Lemma 2.1.2(ii), there exists a non-projective indecomposable kH-module V_0' (unique up to isomorphism) which is a direct summand of V' with $\underline{\mathrm{End}}_{kH}(V_0') = k$ and $R(H,V') \cong R(H,V_0')$. In the situation of Lemma 2.2.2, it then follows that $R(G,V) \cong R(H,V_0')$. In the situation of Lemma 2.2.3, we have at least $R(G,V)/pR(G,V) \cong R(H,V_0')/pR(H,V_0')$

Remark 2.2.5. Let Λ be a finite dimensional k-algebra, and denote by $\mathrm{mod}_{\mathcal{P}}(\Lambda)$ the full subcategory of Λ-mod whose objects are the modules which have no non-zero projective summands. Suppose Λ' is another finite dimensional k-algebra, and $F : \Lambda\text{-}\underline{\mathrm{mod}} \to \Lambda'\text{-}\underline{\mathrm{mod}}$ is a stable equivalence. Let

$$0 \to A \xrightarrow{\binom{f}{s}} B \coprod P \xrightarrow{(g,t)} C \to 0$$

be an almost split sequence in Λ-mod where A, B, C are in $\mathrm{mod}_{\mathcal{P}}(\Lambda)$, B is non-zero and P is projective. Then, by [2, Prop. X.1.6], for any morphism $g' : F(B) \to F(C)$ with $F(\underline{g}) = \underline{g}'$ there is an almost split sequence

$$0 \to F(A) \xrightarrow{\binom{f'}{u}} F(B) \coprod P' \xrightarrow{(g',v)} F(C) \to 0$$

in Λ'-mod where P' is projective and $F(\underline{f}) = \underline{f}'$.

Moreover, by [2, Cor. X.1.9 and Prop. X.1.12], if Λ and Λ' are selfinjective with no blocks of Loewy length 2, then the stable Auslander-Reiten quivers of Λ and Λ' are isomorphic stable translation quivers, and F commutes with Ω.

2.3. **Universal deformation rings that are quotient rings of $W[[t]]$.** In this subsection, we provide a few results that help determine universal deformation rings that are certain quotient rings of $W[[t]]$. As before, let G be a finite group. The first result deals with universal deformation rings modulo p.

Lemma 2.3.1. *Let Y be a finitely generated uniserial kG-module satisfying $\mathrm{End}_{kG}(Y) = k$ and $\mathrm{Ext}^1_{kG}(Y,Y) = k$. Suppose Y has descending radical series $(T_1, T_2, \ldots, T_\ell)$ where $\ell \geq 1$ and $T_1, \ldots, T_\ell$ are simple kG-modules, not necessarily distinct. Assume there exists an integer $s \geq 1$ such that the projective cover P_{T_1} has the form $P_{T_1} = \begin{matrix} & T_1 & \\ U_1 & & U_2 \\ & T_1 & \end{matrix}$ where U_1 and U_2 are uniserial kG-modules, U_1 may be zero, and U_2 has descending radical series*

$$(T_2, \ldots, T_\ell, T_1, T_2, \ldots, T_\ell, \ldots, T_1, T_2, \ldots, T_\ell)$$

of length $\ell p^s - 1$. Define $\overline{U}$ to be the uniserial kG-module $\overline{U} = \begin{matrix} T_1 \\ U_2 \end{matrix}$, and suppose $\mathrm{Ext}^1_{kG}(\overline{U}, Y) = 0$. Then the universal deformation ring of Y modulo p is $\overline{R} = R(G,Y)/pR(G,Y) \cong k[t]/(t^{p^s})$, and the universal mod p deformation of Y over $\overline{R}$ is represented by the kG-module $\overline{U}$.

Proof. By assumption, $\mathrm{Ext}^1_{kG}(Y,Y) = k$, which implies that $\overline{R} \cong k[t]/(t^r)$ for some $r \geq 1$. The module $\overline{U}$ is a uniserial kG-module of length ℓp^s with descending radical series

$$(T_1, \ldots, T_\ell, T_1 \ldots, T_\ell, \ldots, T_1, \ldots, T_\ell) = (T_1, \ldots, T_\ell)^{p^s}.$$

If we let t act as the shift down by ℓ, it follows that $\overline{U}$ is a free $k[t]/(t^{p^s})$-module which is a lift of Y over $k[t]/(t^{p^s})$. Hence there is a k-algebra homomorphism

$$\phi : \overline{R} \to k[t]/(t^{p^s})$$

corresponding to $\overline{U}$. Since $\overline{U}$ is indecomposable as a kG-module, it follows that ϕ is surjective. We now show that ϕ is a k-algebra isomorphism. Suppose this is false. Then there exists a surjective k-algebra homomorphism $\phi_1 : \overline{R} \to k[t]/(t^{p^s+1})$ such that $\pi\phi_1 = \phi$ where $\pi : k[t]/(t^{p^s+1}) \to k[t]/(t^{p^s})$ is the natural projection. Let $\overline{U}_1$ be a lift of Y over $k[t]/(t^{p^s+1})$ relative to ϕ_1. Then $\overline{U}_1$ is a lift of $\overline{U}$ over $k[t]/(t^{p^s+1})$ with $t^{p^s}\overline{U}_1 \cong Y$. Thus we have a short exact sequence of $k[t]/(t^{p^s+1})\,G$-modules

$$0 \to t^{p^s}\overline{U}_1 \to \overline{U}_1 \to \overline{U} \to 0. \tag{2.3.1}$$

We now show that this sequence cannot split as a sequence of kG-modules. Suppose it splits. Then $\overline{U}_1 \cong Y \oplus \overline{U}$ as kG-modules. Let $z = \begin{pmatrix} y \\ u \end{pmatrix} \in Y \oplus \overline{U} \cong \overline{U}_1$. Then t acts on z as multiplication by the matrix

$$A_t = \begin{pmatrix} 0 & \alpha \\ 0 & \mu_t \end{pmatrix}$$

where $\alpha : \overline{U} \to Y$ is a surjective kG-module homomorphism, and μ_t is multiplication by t on $\overline{U}$. Since $t^{p^s}\overline{U}_1 \cong Y$, there exists a non-zero $z = \begin{pmatrix} y \\ u \end{pmatrix} \in Y \oplus \overline{U} \cong \overline{U}_1$ with $(A_t)^{p^s} z \neq 0$. But, since $\mathrm{End}_{kG}(Y) = k$, α corresponds to the isomorphism $\overline{U}/t\overline{U} \cong Y$ which means that the kernel of α is $t\overline{U}$. Thus

$$(A_t)^{p^s} \begin{pmatrix} y \\ u \end{pmatrix} = \begin{pmatrix} \alpha(\mu_t^{p^s-1}(u)) \\ \mu_t^{p^s}(u) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

which gives a contradiction. Hence the short exact sequence (2.3.1) does not split as a sequence of kG-modules. Since $\mathrm{Ext}^1_{kG}(\overline{U}, Y) = 0$ by assumption and $t^{p^s}\overline{U}_1 \cong Y$, this is impossible. Therefore, $\overline{U}_1$ does not exist, which means that ϕ is a k-algebra isomorphism. Thus $\overline{R} \cong k[t]/(t^{p^s})$, and the universal mod p deformation of Y over $\overline{R}$ is represented by the kG-module $\overline{U}$. □

The next result analyzes when a finitely generated kG-module can be lifted to a WG-module which is free as a W-module. In the following, F denotes the fraction field of W.

Lemma 2.3.2. *Let V be a finitely generated kG-module such that there is a non-split short exact sequence of kG-modules*

$$0 \to Y_2 \to V \to Y_1 \to 0 \tag{2.3.2}$$

with $\mathrm{Ext}^1_{kG}(Y_1, Y_2) = k$. Assume that for $i \in \{1, 2\}$, there exists a WG-module X_i which is a lift of Y_i over W; in particular X_i is free as W-module. Suppose that

$$\dim_F \mathrm{Hom}_{FG}(F \otimes_W X_1, F \otimes_W X_2) = \dim_k \mathrm{Hom}_{kG}(Y_1, Y_2) - 1. \tag{2.3.3}$$

Then there exists a WG-module X which is a lift of V over W; in particular, X is free as W-module.

Proof. Since X_1 and X_2 are free as W-modules, we see, using spectral sequences, that

$$\begin{aligned} \mathrm{H}^i(G, \mathrm{Hom}_W(X_1, X_2)) &= \mathrm{Ext}^i_{WG}(X_1, X_2), \text{ and} \\ \mathrm{H}^i(G, \mathrm{Hom}_k(Y_1, Y_2)) &= \mathrm{Ext}^i_{kG}(Y_1, Y_2) \end{aligned}$$

for all $i \geq 0$. From the short exact sequence

$$0 \to \mathrm{Hom}_W(X_1, X_2) \xrightarrow{\cdot p} \mathrm{Hom}_W(X_1, X_2) \xrightarrow{\mathrm{mod}\ p} \mathrm{Hom}_k(Y_1, Y_2) \to 0$$

we obtain a long exact cohomology sequence

$$\begin{array}{l} 0 \longrightarrow \mathrm{Hom}_{WG}(X_1, X_2) \xrightarrow{\mu_*^0} \mathrm{Hom}_{WG}(X_1, X_2) \xrightarrow{\pi_*^0} \mathrm{Hom}_{kG}(Y_1, Y_2) \xrightarrow{\delta^0} \\ \mathrm{Ext}^1_{WG}(X_1, X_2) \xrightarrow{\mu_*^1} \mathrm{Ext}^1_{WG}(X_1, X_2) \xrightarrow{\pi_*^1} \mathrm{Ext}^1_{kG}(Y_1, Y_2) \longrightarrow \cdots \end{array}$$

where for $i \in \{0, 1\}$, μ_*^i stands for multiplication by p and π_*^i stands for reduction modulo p. To prove that there is a lift of V over W, it is enough to prove that π_*^1 is surjective.

Let $n = \dim_F \mathrm{Hom}_{FG}(F \otimes_W X_1, F \otimes_W X_2)$. Since X_1 and X_2 are free W-modules of finite rank, it follows that

$$F \otimes_W \mathrm{Hom}_{WG}(X_1, X_2) \cong \mathrm{Hom}_{FG}(F \otimes_W X_1, F \otimes_W X_2)$$

as F-modules. Thus $\mathrm{Hom}_{WG}(X_1, X_2)$ is a free W-module of rank n. Hence $\mathrm{Im}(\pi_*^0) = k^n$ and thus $\mathrm{Ker}(\delta^0) = k^n$, as W-modules. On the other hand by (2.3.3), $\mathrm{Hom}_{kG}(Y_1, Y_2) = k^{n+1}$. Thus $\mathrm{Im}(\delta^0) = k$, which implies $\mathrm{Ker}(\mu_*^1) = k$. Hence $\mathrm{Ext}^1_{WG}(X_1, X_2)$ is a non-zero finitely generated W-module, and so the image of π_*^1, which is isomorphic to the cokernel of μ_*^1, is non-trivial. Since $\mathrm{Ext}^1_{kG}(Y_1, Y_2) = k$, it follows that π_*^1 is surjective. □

For the remainder of this section, we consider the case $p = 2$ and prove some results which help determine the universal deformation ring R provided $R/2R \cong k[t]/(t^{2^n})$ for some positive integer n. The first result is a generalization of [7, Lemma 2.2].

Lemma 2.3.3. *Suppose k has characteristic $p = 2$. Let $d \geq 3$ be an integer, and let $f(t)$ be a monic polynomial in $W[t]$ of degree $2^{d-2} - 1$ where all non-leading coefficients lie in $2W$. Suppose R is a complete local Noetherian W-algebra with residue field k for which there is a continuous surjection $\tau : R \to W[[t]]/(f(t))$ and an isomorphism $\mu : R/2R \to k[s]/(s^{2^{d-2}})$ of W-algebras. Then R is isomorphic to $W[[t]]/(f(t)(t - 2\gamma), \alpha 2^m f(t))$ as a W-algebra, where $\gamma \in W$, $\alpha \in \{0, 1\}$ and $0 < m \in \mathbb{Z}$.*

Proof. It follows from the assumptions that there is a continuous W-algebra surjection $\psi : W[[t]] \to R$. Then $\tau \circ \psi : W[[t]] \to W[[t]]/(f(t))$ is a surjective W-algebra homomorphism. Hence modulo 2, t is sent to a generator of the maximal ideal of $W[[t]]/(2, f(t)) \cong k[[t]]/(t^{2^{d-2}-1})$, which means that t is sent to $u(t) \cdot t$ for some unit $u(t)$ in $W[[t]]/(2, f(t))$. Since we can lift $u(t)$ to a unit in $W[[t]]$, we see that $\tau(\psi(t)) = v(t) \cdot t + 2b(t)$ for certain $v(t) \in W[[t]]^*$ and $b(t) \in W[[t]]$. Composing ψ with the continuous W-algebra automorphism $W[[t]] \to W[[t]]$ which sends t to $v(t)^{-1}(t - 2b(t))$, we can assume that $\tau(\psi(t)) = t$. Then $\ker(\tau \circ \psi)$ is the ideal $(f(t)) = W[[t]] \cdot f(t)$. Hence the kernel J of ψ is contained in the ideal $(f(t))$. Moreover, J is properly contained in $(f(t))$, since otherwise $J = (f(t))$ and then $R \cong W[[t]]/(f(t))$ as W-algebras. But this is impossible since $R/2R \cong k[s]/(s^{2^{d-2}})$.

The maximal ideal of $W[[t]]$ is generated by 2 and t. So the maximal ideal of $R/2R$ is generated by the image of t under the surjection $W[[t]] \to R/2R$ induced by $\psi : W[[t]] \to R$. However, the isomorphism $\mu : R/2R \to k[s]/(s^{2^{d-2}})$ shows that as a W-module, $R/2R$ is generated by 1 together with the powers $\xi, \xi^2, \ldots, \xi^{2^{d-2}-1}$ for any generator ξ of the maximal ideal of $R/2R$. So $R/2R$ is generated as a W-module by the images of $1, t, t^2, \ldots, t^{2^{d-2}-1}$. Hence the image of $W \oplus Wt \oplus \cdots \oplus Wt^{2^{d-2}-1} \subset W[[t]]$ under $\psi : W[[t]] \to R$ must be all of R since R is complete. Thus $\psi(t^{2^{d-2}}) = \psi(a_0 + a_1 t + \cdots + a_{2^{d-2}-1} t^{2^{d-2}-1})$ for some $a_0, a_1, \ldots, a_{2^{d-2}-1} \in W$. This means that $t^{2^{d-2}} - (a_0 + a_1 t + \cdots + a_{2^{d-2}-1} t^{2^{d-2}-1}) = j \in J$. But $J \subseteq (f(t))$, so

$$t^{2^{d-2}} - (a_0 + a_1 t + \cdots + a_{2^{d-2}-1} t^{2^{d-2}-1}) = f(t) \cdot q(t) \tag{2.3.4}$$

for a unique $q(t) \in W[[t]]$. Let $q(t) = c_0 + t + \sum_{i=1}^{\infty} c_i t^i$. Suppose there exists $i \geq 1$ with $c_i \neq 0$. Letting $r = \min\{\mathrm{ord}_2(c_i) \mid i \geq 1\}$, we can rewrite $q(t) = c_0 + t + 2^r t w(t)$ for some $w(t) \in W[[t]]$ which is not congruent to the zero power series modulo 2. Comparing the coefficients modulo 2^{r+1} of the terms of degree at least 2^{d-2} on both sides of (2.3.4), we see that $2^r w(t)$ must be congruent to the zero power series modulo 2^{r+1}, which is impossible. Hence $q(t) = t + c_0$ for some $c_0 \in W$. Moreover, c_0 is in $2W$, since otherwise $c_0 \in W^*$ and hence $t + c_0 \in W[[t]]^*$. But then $f(t) = (t + c_0)^{-1} j \in J$, which is impossible since we showed that J is properly contained in $(f(t))$. So $c_0 = -2\gamma$ for some $\gamma \in W$.

This means $f(t)(t - 2\gamma)W[[t]] \subseteq J \subset f(t)W[[t]]$. Hence $J = f(t)J'$ where

$$(t - 2\gamma) = (t - 2\gamma)W[[t]] \subseteq J' \subset W[[t]]$$

and $J' \neq W[[t]]$. Therefore, $J'/(t - 2\gamma)$ is a proper ideal of $W[[t]]/(t - 2\gamma) \cong W$, and hence either zero or generated by a positive power of 2. It follows that $J' = (t - 2\gamma, \alpha 2^m)$ where $\alpha \in \{0, 1\}$ and $m \in \mathbb{Z}^+$. Thus $J = (f(t)(t - 2\gamma), \alpha 2^m f(t))$. □

We now specialize to a particular $f(t)$.

Definition 2.3.4. Suppose k has characteristic $p = 2$. Let F be the fraction field of W, and fix an algebraic closure $\overline{F}$ of F. Let $d \geq 3$ be an integer, and let ζ_{2^ℓ} be a fixed primitive 2^ℓ-th root of unity in $\overline{F}$ for $2 \leq \ell \leq d-1$.

i. Define
$$p_d(t) = \prod_{\ell=2}^{d-1} \text{min.pol.}_F(\zeta_{2^\ell} + \zeta_{2^\ell}^{-1}),$$
and let $R' = W[[t]]/(p_d(t))$.

ii. Let $Z = \langle \sigma \rangle$ be a cyclic group of order 2^{d-1}, and let $\tau : Z \to Z$ be the group automorphism sending σ to σ^{-1}. Then τ can be extended to a W-algebra automorphism of the group ring WZ which will again be denoted by τ. Let $T(\sigma^2) = 1 + \sigma^2 + \sigma^4 + \cdots + \sigma^{2^{d-1}-2}$, and define
$$S' = (WZ)^{\langle \tau \rangle} / \left(T(\sigma^2), \sigma T(\sigma^2)\right).$$

Remark 2.3.5. The minimal polynomial $\text{min.pol.}_F(\zeta_{2^\ell} + \zeta_{2^\ell}^{-1})$ for $\ell \geq 2$ is as follows:
$$\begin{aligned}
\text{min.pol.}_F(\zeta_{2^2} + \zeta_{2^2}^{-1})(t) &= t, \\
\text{min.pol.}_F(\zeta_{2^\ell} + \zeta_{2^\ell}^{-1})(t) &= \left(\text{min.pol.}_F(\zeta_{2^{\ell-1}} + \zeta_{2^{\ell-1}}^{-1})(t)\right)^2 - 2 \qquad \text{for } \ell \geq 3.
\end{aligned}$$

The W-algebra R' from Definition 2.3.4 is a complete local Noetherian ring with residue field k. Moreover,
$$\begin{aligned}
F \otimes_W R' &\cong \prod_{\ell=2}^{d-1} F(\zeta_{2^\ell} + \zeta_{2^\ell}^{-1}) \quad \text{as } F\text{-algebras}, \\
k \otimes_W R' &\cong k[t]/(t^{2^{d-2}-1}) \quad \text{as } k\text{-algebras}.
\end{aligned}$$
Additionally, for any sequence $(r_\ell)_{\ell=2}^{d-1}$ of odd integers, R' is isomorphic to the W-subalgebra of
$$\prod_{\ell=2}^{d-1} W[\zeta_{2^\ell} + \zeta_{2^\ell}^{-1}]$$
generated by the element $\left(\zeta_{2^\ell}^{r_\ell} + \zeta_{2^\ell}^{-r_\ell}\right)_{\ell=2}^{d-1}$.

Lemma 2.3.6. *Using the notations of Definition 2.3.4, there is a continuous W-algebra isomorphism $\rho : R' \to S'$ with $\rho(t) = \sigma + \sigma^{-1}$.*

Suppose now that D_{2^d} is a dihedral group of order 2^d. If in Lemma 2.3.3 the polynomial $f(t)$ is taken to be equal to $p_d(t)$, $\alpha = 1$ and $m = 1$, then the ring R in this Lemma is isomorphic to a subquotient algebra of the group ring WD_{2^d}.

Proof. The ring of invariants $(WZ)^{\langle \tau \rangle}$ is a free W-module with basis
$$\{1, \sigma + \sigma^{-1}, \sigma^2 + \sigma^{-2}, \ldots, \sigma^{2^{d-2}-1} + \sigma^{-2^{d-2}+1}, \sigma^{2^{d-2}}\}.$$
Hence the W-rank of this module is $2^{d-2} + 1$. Expanding expressions of the form $(\sigma + \sigma^{-1})^\ell$ for various ℓ, one sees that as W-algebra $(WZ)^{\langle \tau \rangle}$ is generated by $(\sigma + \sigma^{-1})$ and $\sigma^{2^{d-2}}$. In S', the residue class of $\sigma^{2^{d-2}}$ can be expressed as a polynomial in $(\sigma + \sigma^{-1})$, which means that S' is generated as a W-algebra by the residue class of $(\sigma + \sigma^{-1})$. Since the residue class of $(\sigma^{2^{d-2}-1} + \sigma^{-2^{d-2}+1})$ can also be expressed as a polynomial in $(\sigma + \sigma^{-1})$ in S' and since S' has no torsion, we conclude that S' is a free W-module of rank $2^{d-2} + 1 - 2 = 2^{d-2} - 1$.

Define $\hat{\rho} : W[[t]] \to S'$ to be the continuous W-algebra homomorphism sending t to the residue class of $(\sigma + \sigma^{-1})$. Then $\hat{\rho}$ is surjective. Using the description of the minimal polynomial of $(\zeta_{2^\ell} + \zeta_{2^\ell}^{-1})$, $2 \leq \ell \leq d-1$, over F in Remark 2.3.5, we see that
$$[\text{min.pol.}_F(\zeta_{2^\ell} + \zeta_{2^\ell}^{-1})]\,(\sigma + \sigma^{-1}) = \sigma^{2^{\ell-2}} + \sigma^{-2^{\ell-2}}.$$

Hence we have

$$p_d(\sigma+\sigma^{-1}) = \prod_{\ell=2}^{d-1}(\sigma^{2^{\ell-2}}+\sigma^{-2^{\ell-2}}), \tag{2.3.5}$$

and we see by induction that the latter is equal to

$$(\sigma+\sigma^{-1})+(\sigma^3+\sigma^{-3})+\cdots+(\sigma^{2^{d-2}-1}+\sigma^{-2^{d-2}+1}) = \sigma T(\sigma^2) \tag{2.3.6}$$

which is zero in S'. Thus $p_d(t)$ lies in the kernel of $\hat{\rho}$. This means that we obtain a surjective continuous W-algebra homomorphism

$$\rho : R' = W[[t]]/(p_d(t)) \to S'.$$

Since both R' and S' are free over W of rank $2^{d-2}-1$, it follows that ρ is a continuous W-algebra isomorphism. In particular, R' is isomorphic to a subquotient algebra of the group ring WD_{2^d} when D_{2^d} is a dihedral group of order 2^d.

Suppose now that in Lemma 2.3.3 the polynomial $f(t)$ is taken to be equal to $p_d(t)$, $\alpha=1$ and $m=1$. Then the ring R in this Lemma is isomorphic to $W[[t]]/(p_d(t)(t-2), 2\,p_d(t))$ as a W-algebra. Thus to finish the proof of Lemma 2.3.6, it suffices to show that the ring $W[[t]]/(p_d(t)(t-2))$ is isomorphic to a subquotient algebra of WD_{2^d}. Define

$$\Theta = (WZ)^{\langle\tau\rangle}/\left(T(\sigma^2)-\sigma T(\sigma^2)\right).$$

Then Θ is isomorphic to a subquotient algebra of WD_{2^d} and it is generated as a W-algebra by the residue class of $(\sigma+\sigma^{-1})$. Moreover, Θ is a free W-module of rank 2^{d-2}, since the ideal $\left(T(\sigma^2)-\sigma T(\sigma^2)\right)$ is generated over W by $T(\sigma^2)-\sigma T(\sigma^2)$. Define a continuous W-algebra homomorphism $\theta : W[[t]] \to \Theta$ by sending t to the residue class of $(\sigma+\sigma^{-1})$. Then, using (2.3.5) and (2.3.6), we see that

$$\begin{aligned}\theta(p_d(t)(t-2)) &= p_d(\sigma+\sigma^{-1})((\sigma+\sigma^{-1})-2)\\ &= \sigma T(\sigma^2)((\sigma+\sigma^{-1})-2)\\ &= 2\left[T(\sigma^2)-\sigma T(\sigma^2)\right]\end{aligned}$$

which is zero in Θ. Hence $p_d(t)(t-2)$ lies in the kernel of θ. This means that we obtain a surjective continuous W-algebra homomorphism

$$\overline{\theta} : W[[t]]/(p_d(t)(t-2)) \to \Theta.$$

Since both $W[[t]]/(p_d(t)(t-2))$ and Θ are free over W of rank 2^{d-2}, it follows that $\overline{\theta}$ is a continuous W-algebra isomorphism. Thus $W[[t]]/(p_d(t)(t-2))$ is isomorphic to a subquotient algebra of WD_{2^d}, which completes the proof of Lemma 2.3.6. □

3. Blocks with dihedral defect groups

Let k be an algebraically closed field of characteristic $p=2$, and let W be the ring of infinite Witt vectors over k. Let G be a finite group, and let B be a block of kG with dihedral defect groups which is Morita equivalent to the principal block of a finite simple group. From the classification by Gorenstein and Walter of the groups with dihedral Sylow 2-subgroups in [22], it follows that there are three families of such blocks B, up to Morita equivalence:

i. the principal 2-modular blocks of $\mathrm{PSL}_2(\mathbb{F}_q)$ where $q \equiv 1 \mod 4$,
ii. the principal 2-modular blocks of $\mathrm{PSL}_2(\mathbb{F}_q)$ where $q \equiv 3 \mod 4$, and
iii. the principal 2-modular block of the alternating group A_7.

Note that in all cases (i) - (iii), B contains precisely 3 isomorphism classes of simple modules.

Remark 3.1. In [20], Erdmann classified all blocks with dihedral defect groups. It follows that if we consider all such blocks B_0 containing precisely 3 isomorphism classes of simple modules, there is one more family attached to case (iii) containing a Morita equivalence class of possible blocks for

each defect $d \geq 3$. By [20, §X.4], the blocks in this family having defect $d \geq 4$ cannot be excluded as possible blocks with dihedral defect groups.

However, if we assume that B_0 is Morita equivalent to the principal block of kH for some finite (not necessarily simple) group H, we can exclude this family as follows. Since k has characteristic 2, we can assume that H has no normal subgroup of odd order. By [22], it then follows that H is isomorphic to either a subgroup of $\mathrm{P\Gamma L}_2(\mathbb{F}_q)$ containing $\mathrm{PSL}_2(\mathbb{F}_q)$ for some odd prime power q, or to the alternating group A_7. Using a theorem by Clifford [23, Hauptsatz V.17.3], we see that the only possibility for B_0 to be Morita equivalent to a block in the bigger family attached to case (iii) occurs when B_0 has defect $d = 3$, i.e. B_0 is Morita equivalent to the principal 2-modular block of the alternating group A_7.

The blocks in (i), (ii) and (iii) are all Morita equivalent to basic algebras of special biserial algebras. (For the relevant background on special biserial algebras we refer to §7.) In §3.1, §3.2 and §3.3, we give the quivers and relations for the basic algebras of these blocks, together with their projective indecomposable modules and their decomposition matrices. In §3.4, we then state some results from [9] about the ordinary irreducible characters of G which belong to B.

3.1. **The principal 2-modular block of** $\mathrm{PSL}_2(\mathbb{F}_q)$ **when** $q \equiv 1 \mod 4$**.** Let G be a finite group, and let B be a block of kG which is Morita equivalent to the principal block of $k\mathrm{PSL}_2(\mathbb{F}_q)$ where $q \equiv 1 \mod 4$. Suppose that 2^d is the order of the defect groups of B, i.e. the order of the Sylow 2-subgroups of $\mathrm{PSL}_2(\mathbb{F}_q)$. Then, by [20], B is Morita equivalent to the special biserial algebra $\Lambda = kQ/I$ where Q is given in Figure 3.1.1 and

$$I = \langle \gamma\beta, \delta\eta, (\eta\delta\beta\gamma)^{2^{d-2}} - (\beta\gamma\eta\delta)^{2^{d-2}} \rangle.$$

We denote the irreducible Λ-modules by S_0, S_1, S_2, or, using short-hand, by $0, 1, 2$. The radical series of the projective indecomposable Λ-modules (and hence of the projective indecomposable B-modules) are described in Figure 3.1.2 where the radical series length of each of these modules is $2^d + 1$. The decomposition matrix of B is given in Figure 3.1.3.

FIGURE 3.1.1. The quiver Q for blocks as in §3.1.

$$Q = \begin{tikzcd} 1 \bullet \arrow[r, shift left, "\beta"] & \overset{0}{\bullet} \arrow[l, shift left, "\gamma"] \arrow[r, shift left, "\delta"] & \bullet\, 2 \arrow[l, shift left, "\eta"] \end{tikzcd}$$

FIGURE 3.1.2. The radical series of the projective indecomposable modules for blocks as in §3.1.

$$P_0 = \begin{array}{ccc} & 0 & \\ 1 & & 2 \\ 0 & & 0 \\ 2 & & 1 \\ 0 & & 0 \\ \vdots & & \vdots \\ 0 & & 0 \\ 1 & & 2 \\ 0 & & 0 \\ 2 & & 1 \\ & 0 & \end{array}, \qquad P_1 = \begin{array}{c} 1 \\ 0 \\ 2 \\ 0 \\ 1 \\ \vdots \\ 0 \\ 2 \\ 0 \\ 1 \end{array}, \qquad P_2 = \begin{array}{c} 2 \\ 0 \\ 1 \\ 0 \\ 2 \\ \vdots \\ 0 \\ 1 \\ 0 \\ 2 \end{array}.$$

FIGURE 3.1.3. The decomposition matrix for blocks as in §3.1.

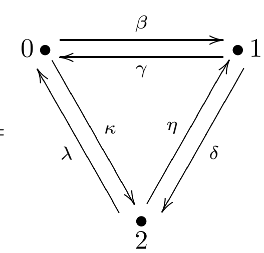
$$\begin{array}{c} \\ \chi_1 \\ \chi_2 \\ \chi_3 \\ \chi_4 \\ \chi_{5,i} \end{array} \begin{array}{c} \begin{array}{ccc} \varphi_0 & \varphi_1 & \varphi_2 \end{array} \\ \left[\begin{array}{ccc} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \\ 2 & 1 & 1 \end{array}\right] \end{array} \quad 1 \le i \le 2^{d-2}-1.$$

3.2. **The principal 2-modular block of** $\mathrm{PSL}_2(\mathbb{F}_q)$ **when** $q \equiv 3 \mod 4$**.** Let G be a finite group, and let B be a block of kG which is Morita equivalent to the principal block of $k\mathrm{PSL}_2(\mathbb{F}_q)$ where $q \equiv 3 \mod 4$. Suppose that 2^d is the order of the defect groups of B. Then, by [20], B is Morita equivalent to the special biserial algebra $\Lambda = kQ/I$ where Q is given in Figure 3.2.1 and

$$I = \langle \delta\beta, \lambda\delta, \beta\gamma, \kappa\gamma, \eta\kappa, \gamma\eta, \gamma\beta - \lambda\kappa, \kappa\lambda - (\delta\eta)^{2^{d-2}}, (\eta\delta)^{2^{d-2}} - \beta\gamma \rangle.$$

We denote the irreducible Λ-modules by S_0, S_1, S_2, or, using short-hand, by $0, 1, 2$. The radical

FIGURE 3.2.1. The quiver Q for blocks as in §3.2.

$Q =$ 

series of the projective indecomposable Λ-modules (and hence of the projective indecomposable B-modules) are described in Figure 3.2.2 where for $i \in \{1, 2\}$, $\mathrm{rad}(P_i)/\mathrm{soc}(P_i)$ is isomorphic to the

FIGURE 3.2.2. The radical series of the projective indecomposable modules for blocks as in §3.2.

$$P_0 = \begin{array}{ccc} & 0 & \\ 1 & & 2 \\ & 0 & \end{array}, \qquad P_1 = \begin{array}{ccc} & 1 & \\ 0 & & 2 \\ & & 1 \\ & & 2 \\ & & \vdots \\ & & 1 \\ & & 2 \\ & 1 & \end{array}, \qquad P_2 = \begin{array}{ccc} & 2 & \\ 0 & & 1 \\ & & 2 \\ & & 1 \\ & & \vdots \\ & & 2 \\ & & 1 \\ & 2 & \end{array}.$$

direct sum of S_0 and a uniserial module of length $2^{d-1} - 1$. The decomposition matrix of B is given in Figure 3.2.3.

Figure 3.2.3. The decomposition matrix for blocks as in §3.2.

$$\begin{array}{c c}
 & \begin{array}{ccc} \varphi_0 & \varphi_1 & \varphi_2 \end{array} \\
\begin{array}{c} \chi_1 \\ \chi_2 \\ \chi_3 \\ \chi_4 \\ \chi_{5,i} \end{array} & \left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{array}\right] \quad 1 \le i \le 2^{d-2}-1.
\end{array}$$

3.3. **The principal 2-modular block of A_7.** Let G be a finite group, and let B be a block of kG which is Morita equivalent to the principal block of kA_7. Suppose that 2^d is the order of the defect groups of B. Then $d = 3$, and by [20], B is Morita equivalent to the special biserial algebra $\Lambda = kQ/I$ where Q is given in Figure 3.3.1 and

$$I = \langle \beta\alpha, \alpha\gamma, \gamma\beta, \delta\eta, \eta\delta\beta\gamma - \beta\gamma\eta\delta, \alpha^2 - \gamma\eta\delta\beta \rangle.$$

We denote the irreducible Λ-modules by S_0, S_1, S_2, or, using short-hand, by $0, 1, 2$. The radical

Figure 3.3.1. The quiver Q for blocks as in §3.3.

$$Q = \begin{tikzcd} \overset{1}{\bullet} \arrow[loop left, "\alpha"] \arrow[r, shift left, "\beta"] & \overset{0}{\bullet} \arrow[l, shift left, "\gamma"] \arrow[r, shift left, "\delta"] & \bullet\, 2 \arrow[l, shift left, "\eta"] \end{tikzcd}$$

series of the projective indecomposable Λ-modules (and hence of the projective indecomposable B-modules) are described in Figure 3.3.2 where $\mathrm{rad}(P_1)/\mathrm{soc}(P_1)$ is isomorphic to the direct sum of

Figure 3.3.2. The radical series of the projective indecomposable modules for blocks as in §3.3.

$$P_0 = \begin{array}{ccc} & 0 & \\ 1 & & 2 \\ 0 & & 0 \\ 2 & & 1 \\ & 0 & \end{array}, \qquad P_1 = \begin{array}{ccc} & 1 & \\ 1 & & 0 \\ & & 2 \\ & & 0 \\ & 1 & \end{array}, \qquad P_2 = \begin{array}{c} 2 \\ 0 \\ 1 \\ 0 \\ 2 \end{array}.$$

S_1 and a uniserial module with radical series (S_0, S_2, S_0). The decomposition matrix of B is given in Figure 3.3.3.

3.4. **Ordinary characters for blocks with dihedral defect groups.** Let G be a finite group and let B be a block of kG with dihedral defect group D of order 2^d where $d \ge 3$. Moreover, assume that B contains exactly three isomorphism classes of simple kG-modules. This means that in the notation of [9, §4] we are in Case (aa) (see [9, Thm. 2]).

Let F be the fraction field of W, and let ζ_{2^ℓ} be a fixed primitive 2^ℓ-th root of unity in an algebraic closure of F for $2 \le \ell \le d-1$. Let

$$\chi_1, \chi_2, \chi_3, \chi_4, \qquad \chi_{5,i}, 1 \le i \le 2^{d-2}-1,$$

FIGURE 3.3.3. The decomposition matrix for blocks as in §3.3.

$$\begin{array}{c} \\ \chi_1 \\ \chi_2 \\ \chi_3 \\ \chi_4 \\ \chi_{5,i} \end{array} \begin{array}{c} \begin{array}{ccc} \varphi_0 & \varphi_1 & \varphi_2 \end{array} \\ \left[\begin{array}{ccc} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{array}\right] \end{array} \quad i = 1.$$

be the ordinary irreducible characters of G belonging to B. Let σ be an element of order 2^{d-1} in D. By [9], there is a block b_σ of $kC_G(\sigma)$ with $b_\sigma^G = B$ which contains a unique 2-modular character $\varphi^{(\sigma)}$ such that the following is true. There is an ordering of $(1, 2, \ldots, 2^{d-2} - 1)$ such that for $1 \le i \le 2^{d-2} - 1$ and r odd,

$$\chi_{5,i}(\sigma^r) = (\zeta_{2^{d-1}}^{ri} + \zeta_{2^{d-1}}^{-ri}) \cdot \varphi^{(\sigma)}(1). \tag{3.4.1}$$

Note that W contains all roots of unity of order not divisible by 2. Hence by [9] and by [21], the characters $\chi_1, \chi_2, \chi_3, \chi_4$ correspond to simple FG-modules. On the other hand, the characters $\chi_{5,i}$, $i = 1, \ldots, 2^{d-2} - 1$, fall into $d - 2$ Galois orbits $\mathcal{O}_2, \ldots, \mathcal{O}_{d-1}$ under the action of $\mathrm{Gal}(F(\zeta_{2^{d-1}} + \zeta_{2^{d-1}}^{-1})/F)$. Namely for $2 \le \ell \le d - 1$, $\mathcal{O}_\ell = \{\chi_{5,2^{d-1-\ell}(2u-1)} \mid 1 \le u \le 2^{\ell-2}\}$. The field generated by the character values of each $\xi_\ell \in \mathcal{O}_\ell$ over F is $F(\zeta_{2^\ell} + \zeta_{2^\ell}^{-1})$. Hence by [21], each ξ_ℓ corresponds to an absolutely irreducible $F(\zeta_{2^\ell} + \zeta_{2^\ell}^{-1})G$-module X_ℓ. By [23, Satz V.14.9], this implies that for $2 \le \ell \le d - 1$, the Schur index of each $\xi_\ell \in \mathcal{O}_\ell$ over F is 1. Hence we obtain $d - 2$ non-isomorphic simple FG-modules $V_2, \ldots, V_{d-1}$ with characters $\rho_2, \ldots, \rho_{d-1}$ satisfying

$$\rho_\ell = \sum_{\xi_\ell \in \mathcal{O}_\ell} \xi_\ell = \sum_{u=1}^{2^{\ell-2}} \chi_{5,2^{d-1-\ell}(2u-1)} \qquad \text{for } 2 \le \ell \le d - 1. \tag{3.4.2}$$

By [23, Hilfssatz V.14.7], $\mathrm{End}_{FG}(V_\ell)$ is a commutative F-algebra isomorphic to the field generated over F by the character values of any $\xi_\ell \in \mathcal{O}_\ell$. This means

$$\mathrm{End}_{FG}(V_\ell) \cong F(\zeta_{2^\ell} + \zeta_{2^\ell}^{-1}) \qquad \text{for } 2 \le \ell \le d - 1. \tag{3.4.3}$$

By [9], the characters $\chi_{5,i}$ have the same degree x for $1 \le i \le 2^{d-2} - 1$. The characters $\chi_1, \chi_2, \chi_3, \chi_4$ have height 0 and $\chi_{5,i}$, $1 \le i \le 2^{d-2} - 1$, have height 1. Hence $x = 2^{a-d+1}x^*$ where $\#G = 2^a \cdot g^*$ and x^* and g^* are odd. Since the centralizer $C_G(\sigma)$ contains $\langle\sigma\rangle$, we have $\#C_G(\sigma) = 2^{d-1} \cdot 2^b \cdot m^*$ where $b \ge 0$ and m^* is odd. Suppose $\varphi^{(\sigma)}(1) = 2^c \cdot n^*$ where $c \ge 0$ and n^* is odd. Note that if ψ is an ordinary irreducible character of $C_G(\sigma)$ belonging to the block b_σ, then by [30, p. 61], $\psi(1)$ divides $(\#C_G(\sigma))/(\#\langle\sigma\rangle) = 2^b \cdot m^*$. Because $\psi(1) = s_\psi \cdot \varphi^{(\sigma)}(1)$ for some positive integer s_ψ, we have $c \le b$.

Let C be the conjugacy class in G of σ, and let $t(C) \in WG$ be the class sum of C. We want to determine the action of $t(C)$ on V_ℓ for $2 \le \ell \le d-1$. For this, we identify $\mathrm{End}_{FG}(V_\ell) \cong F(\zeta_{2^\ell} + \zeta_{2^\ell}^{-1})$ with $\mathrm{End}_{F(\zeta_{2^\ell}+\zeta_{2^\ell}^{-1})G}(X_\ell)$ for one particular absolutely irreducible $F(\zeta_{2^\ell} + \zeta_{2^\ell}^{-1})G$-constituent X_ℓ of V_ℓ with character ξ_ℓ. By (3.4.2), we can choose $\xi_\ell = \chi_{5,2^{d-1-\ell}}$. Then, under this identification, for $2 \le \ell \le d - 1$, the action of $t(C)$ on V_ℓ is given as multiplication by

$$\begin{aligned} \frac{\#C}{\xi_\ell(1)} \cdot \xi_\ell(\sigma) &= \frac{\#C}{\xi_\ell(1)} \cdot \varphi^{(\sigma)}(1) \cdot (\zeta_{2^{d-1}}^{2^{d-1-\ell}} + \zeta_{2^{d-1}}^{-2^{d-1-\ell}}) \\ &= \frac{[G : C_G(\sigma)]}{x} \cdot \varphi^{(\sigma)}(1) \cdot (\zeta_{2^{d-1}}^{2^{d-1-\ell}} + \zeta_{2^{d-1}}^{-2^{d-1-\ell}}) \\ &= 2^{c-b} \frac{g^* \cdot n^*}{m^* \cdot x^*} \cdot (\zeta_{2^{d-1}}^{2^{d-1-\ell}} + \zeta_{2^{d-1}}^{-2^{d-1-\ell}}) \end{aligned} \tag{3.4.4}$$

where, as shown above, $c \le b$. Note that $\frac{g^*\cdot n^*}{m^*\cdot x^*}$ is a unit in W, since $g^*\cdot n^*$ and $m^*\cdot x^*$ are odd. Since $t(C)\in WG$, we must have $c\ge b$, i.e. $c=b$. Therefore, (3.4.4) implies that there exists a unit ω in W such that for $2\le \ell\le d-1$, the action of $t(C)$ on V_ℓ is given as multiplication by

$$\omega\cdot(\zeta_{2^{d-1}}^{2^{d-1-\ell}}+\zeta_{2^{d-1}}^{-2^{d-1-\ell}}) \tag{3.4.5}$$

when we identify $\mathrm{End}_{FG}(V_\ell)$ with $\mathrm{End}_{F(\zeta_{2^\ell}+\zeta_{2^\ell}^{-1})G}(X_\ell)$ for an absolutely irreducible $F(\zeta_{2^\ell}+\zeta_{2^\ell}^{-1})G$-constituent X_ℓ of V_ℓ with character $\chi_{5,2^{d-1-\ell}}$.

4. Universal deformation rings modulo 2

As in §3, let k be an algebraically closed field of characteristic $p=2$, let G be a finite group, and let B be a block of kG with dihedral defect groups containing precisely three isomorphism classes of simple kG-modules. Suppose 2^d is the order of the defect groups of B. In this section, we determine the universal deformation ring modulo 2 for all finitely generated B-modules with stable endomorphism ring k. Since the case $d=2$ has been done in [4], we assume throughout this section that $d\ge 3$.

In §4.1, we first look at blocks B that are Morita equivalent to blocks as in §3.1. In §4.2, we then show how stable equivalences of Morita type can be used to get analogous results for blocks B that are Morita equivalent to blocks as in §3.2 and §3.3.

4.1. **Universal deformation rings modulo 2 for blocks as in §3.1.** The objective of this subsection is to prove the following result:

Proposition 4.1.1. *Let B be a block of kG which is Morita equivalent to $\Lambda = kQ/I$ with Q and I as in §3.1. Suppose 2^d is the order of the defect groups of B with $d\ge 3$. Denote the three simple B-modules by T_0, T_1 and T_2, where T_i corresponds to S_i, for $i\in\{0,1,2\}$, under the Morita equivalence. Let $\mathfrak{C}$ be a component of the stable Auslander-Reiten quiver of B containing a module with endomorphism ring k. Then $\mathfrak{C}$ contains a simple module, or a uniserial module of length 4.*

i. *Suppose $\mathfrak{C}$ contains T_0. Then $\mathfrak{C}$ and $\Omega(\mathfrak{C})$ are both of type $\mathbb{Z}A_\infty^\infty$. All modules M in $\mathfrak{C}\cup\Omega(\mathfrak{C})$ have stable endomorphism ring equal to k and $R(G,M)/2R(G,M)\cong k$.*
ii. *Let $i\in\{1,2\}$, and suppose $\mathfrak{C}$ contains T_i. Then $\mathfrak{C}$ is a 3-tube with T_i belonging to its boundary, and $\mathfrak{C}=\Omega(\mathfrak{C})$. If M lies in $\mathfrak{C}$ having stable endomorphism ring k, then $M\in\{T_i,\Omega^2(T_i),\Omega^4(T_i)\}$ up to isomorphism, and $R(G,M)/2R(G,M)\cong k$.*
iii. *Suppose $\mathfrak{C}$ contains a uniserial module Y of length 4. Then $\mathfrak{C}$ and $\Omega(\mathfrak{C})$ are both of type $\mathbb{Z}A_\infty^\infty$, and $\mathfrak{C}=\Omega(\mathfrak{C})$ exactly when $d=3$. If M lies in $\mathfrak{C}\cup\Omega(\mathfrak{C})$ having stable endomorphism ring k, then M is isomorphic to $\Omega^j(Y)$ for some integer j, and $R(G,M)/2R(G,M)\cong k[t]/(t^{2^{d-2}})$.*

The only components of the stable Auslander-Reiten quiver of B containing modules with stable endomorphism ring k are the ones in $(i)-(iii)$.

Remark 4.1.2. If B is as in Proposition 4.1.1, then there are precisely four uniserial B-modules of length 4:

$$Y_1=\begin{matrix}T_1\\T_0\\T_2\\T_0\end{matrix}\;,\;\Omega^2(Y_1)=\begin{matrix}T_0\\T_2\\T_0\\T_1\end{matrix}\;,\;Y_2=\begin{matrix}T_2\\T_0\\T_1\\T_0\end{matrix}\;,\;\Omega^2(Y_2)=\begin{matrix}T_0\\T_1\\T_0\\T_2\end{matrix}\;.$$

To prove Proposition 4.1.1, we need several Lemmas.

Lemma 4.1.3. *Let $\Lambda=kQ/I$ with Q and I as in §3.1. Let $\mathfrak{C}_0$ be the component of the stable Auslander-Reiten quiver of Λ containing S_0. Let M be an indecomposable Λ-module with $\mathrm{End}_\Lambda(M)=k$. Then M either lies in $\mathfrak{C}_0$, or M is isomorphic to S_1 or S_2, or M is uniserial of length 4.*

Proof. Suppose first that S_0 is a direct summand of $\mathrm{top}(M)$. Then S_0 cannot be a direct summand of $\mathrm{soc}(M)$. The modules S_0, $\begin{smallmatrix}0\\1\end{smallmatrix}$, $\begin{smallmatrix}0\\2\end{smallmatrix}$, $\begin{smallmatrix}0\\1\\0\\2\end{smallmatrix}$, $\begin{smallmatrix}0\\2\\0\\1\end{smallmatrix}$ and $\begin{smallmatrix}&0&\\1&&2\end{smallmatrix}$ have endomorphism ring equal to k. If M is uniserial of length at least 5 or M is not uniserial of length at least 4, then there is an endomorphism of M factoring non-trivially through either S_0, $\begin{smallmatrix}0\\1\end{smallmatrix}$ or $\begin{smallmatrix}0\\2\end{smallmatrix}$, which is a contradiction.

Similarly, we see that if S_0 is a direct summand of $\mathrm{soc}(M)$, then S_0 cannot be a direct summand of $\mathrm{top}(M)$ and M must be isomorphic to S_0, $\begin{smallmatrix}1\\0\end{smallmatrix}$, $\begin{smallmatrix}2\\0\end{smallmatrix}$, $\begin{smallmatrix}2\\0\\1\\0\end{smallmatrix}$, $\begin{smallmatrix}1\\0\\2\\0\end{smallmatrix}$ or $\begin{smallmatrix}1&&2\\&0&\end{smallmatrix}$.

If S_0 is neither a direct summand of $\mathrm{top}(M)$ nor of $\mathrm{soc}(M)$, then M can only be isomorphic to S_1, S_2, $\begin{smallmatrix}1\\0\\2\end{smallmatrix}$ or $\begin{smallmatrix}2\\0\\1\end{smallmatrix}$, since P_1 and P_2 are both uniserial.

Of all the above possibilities for M, only S_1, S_2, and the uniserial modules of length 4 do not lie in $\mathfrak{C}_0$. □

The proofs of the following three Lemmas are rather technical and will be deferred to §6. Let $\Lambda = kQ/I$ with Q and I as in §3.1.

Lemma 4.1.4. *Let $\mathfrak{C}_0$ be the component of the stable Auslander-Reiten quiver of Λ containing S_0, and let M be a Λ-module belonging to $\mathfrak{C}_0 \cup \Omega(\mathfrak{C}_0)$. Then $\mathfrak{C}_0$ and $\Omega(\mathfrak{C}_0)$ are both of type $\mathbb{Z}A_\infty^\infty$, and $\underline{\mathrm{End}}_\Lambda(M) = k$ and $\mathrm{Ext}^1_\Lambda(M,M) = 0$.*

Lemma 4.1.5. *Let $i \in \{1,2\}$ and let $\mathfrak{C}_i$ be the component of the stable Auslander-Reiten quiver of Λ containing S_i. Then $\mathfrak{C}_i$ is a 3-tube with S_i belonging to its boundary, and $\mathfrak{C}_i = \Omega(\mathfrak{C}_i)$. If M belongs to $\mathfrak{C}_i$ and has stable endomorphism ring k, then M is in the Ω-orbit of S_i and $\mathrm{Ext}^1_\Lambda(M,M) = 0$.*

Lemma 4.1.6. *Let $\mathfrak{C}$ be a component of the stable Auslander-Reiten quiver of Λ containing a uniserial module X of length 4. Then $\mathfrak{C}$ and $\Omega(\mathfrak{C})$ are both of type $\mathbb{Z}A_\infty^\infty$, and $\mathfrak{C} = \Omega(\mathfrak{C})$ exactly when $d = 3$. If M belongs to $\mathfrak{C} \cup \Omega(\mathfrak{C})$ and has stable endomorphism ring k, then M is in the Ω-orbit of X and $\mathrm{Ext}^1_\Lambda(M,M) = k$.*

Proof of Proposition 4.1.1. The last statement of Proposition 4.1.1 will also be proved in §6. We now prove the remaining statements. Let $\mathfrak{C}$ be a component of the stable Auslander-Reiten quiver of B containing a module with endomorphism ring k. By Lemma 4.1.3, $\mathfrak{C}$ contains a simple module or a uniserial module of length 4. Part (i) (resp. part (ii)) of Proposition 4.1.1 follows from Lemma 4.1.4 (resp. Lemma 4.1.5). Because of Lemma 4.1.6, Remark 4.1.2 and Lemma 2.1.2, to prove part (iii) we only need to show $R(G,M)/2R(G,M) \cong k[t]/(t^{2^{d-2}})$ if M is either Y_1 or Y_2. We show this for $M = Y_1$. (The case $M = Y_2$ is proved similarly.) Let $\overline{R} = R(G,Y_1)/2R(G,Y_1)$. By Lemmas 4.1.3 and 4.1.6, we have $\mathrm{End}_{kG}(Y_1) = k$ and $\mathrm{Ext}^1_{kG}(Y_1,Y_1) = k$. The projective indecomposable kG-module P_{T_1} has the form $\begin{smallmatrix}T_1\\U\\T_1\end{smallmatrix}$ where U is uniserial of length $4\cdot 2^{d-2}-1$ with descending radical series

$$(T_0, T_2, T_0, T_1, T_0, T_2, T_0, \ldots, T_1, T_0, T_2, T_0).$$

If $\overline{U} = \begin{smallmatrix}T_1\\U\end{smallmatrix}$, then

$$\mathrm{Ext}^1_{kG}(\overline{U}, Y_1) = \underline{\mathrm{Hom}}_{kG}(\Omega(\overline{U}), Y_1) = \underline{\mathrm{Hom}}_{kG}(T_1, Y_1) = 0.$$

By Lemma 2.3.1, this implies $\overline{R} \cong k[t]/(t^{2^{d-2}})$, and the universal mod 2 deformation of Y over $\overline{R}$ is represented by the kG-module $\overline{U}$.

Remark 4.1.7. It follows from the proof of Proposition 4.1.1 that if $Y = \begin{matrix} T_i \\ T_0 \\ T_j \\ T_0 \end{matrix}$ where $i \neq j$ in $\{1, 2\}$, then the universal mod 2 deformation of Y is represented by the uniserial kG-module $\overline{U}$ of length $4 \cdot 2^{d-2}$ with descending radical series

$$(T_i, T_0, T_j, T_0, T_i, T_0, T_j, T_0, \ldots, T_i, T_0, T_j, T_0).$$

Moreover the uniserial kG-module $\overline{U'} \cong \overline{U}/Y$ of length $4 \cdot (2^{d-2} - 1)$ defines a lift of Y over $k[t]/(t^{2^{d-2}-1})$.

4.2. **Universal deformation rings modulo 2 for blocks as in §3.2 and §3.3.** In this subsection we prove analogous results to Proposition 4.1.1 for blocks that are Morita equivalent to blocks as in §3.2 and §3.3. We start with blocks as in §3.2.

Proposition 4.2.1. *Let B be a block of kG which is Morita equivalent to $\Lambda = kQ/I$ with Q and I as in §3.2. Suppose 2^d is the order of the defect groups of B with $d \geq 3$. Denote the three simple B-modules by T_0, T_1 and T_2, where T_i corresponds to S_i, for $i \in \{0, 1, 2\}$, under the Morita equivalence. Let $\mathfrak{C}$ be a component of the stable Auslander-Reiten quiver of B containing a module with endomorphism ring k. Then $\mathfrak{C}$ contains a simple module, or a uniserial module of length 2.*

i. *Suppose $\mathfrak{C}$ contains T_0. Then $\Omega(\mathfrak{C})$ contains T_1 and T_2, and $\mathfrak{C}$ and $\Omega(\mathfrak{C})$ are both of type $\mathbb{Z}A_\infty^\infty$. All modules M in $\mathfrak{C} \cup \Omega(\mathfrak{C})$ have stable endomorphism ring equal to k and $R(G, M)/2R(G, M) \cong k$.*
ii. *Let $i \neq j$ be in $\{1, 2\}$, and suppose $\mathfrak{C}$ contains $T_{0,i} = \begin{matrix} T_0 \\ T_i \end{matrix}$. Then $\mathfrak{C}$ is a 3-tube with $T_{0,i}$ and $T_{j,0} = \begin{matrix} T_j \\ T_0 \end{matrix}$ belonging to its boundary, and $\mathfrak{C} = \Omega(\mathfrak{C})$. If M lies in $\mathfrak{C}$ having stable endomorphism ring k, then $M \in \{T_{0,i}, \Omega^2(T_{0,i}), \Omega^4(T_{0,i})\}$ up to isomorphism, and $R(G, M)/2R(G, M) \cong k$.*
iii. *Suppose $Y \in \left\{ \begin{matrix} T_1 \\ T_2 \end{matrix}, \begin{matrix} T_2 \\ T_1 \end{matrix} \right\}$ and $\mathfrak{C}$ contains Y. Then $\mathfrak{C}$ and $\Omega(\mathfrak{C})$ are both of type $\mathbb{Z}A_\infty^\infty$, and $\mathfrak{C} = \Omega(\mathfrak{C})$ exactly when $d = 3$. If M lies in $\mathfrak{C} \cup \Omega(\mathfrak{C})$ having stable endomorphism ring k, then M is isomorphic to $\Omega^j(Y)$ for some integer j, and $R(G, M)/2R(G, M) \cong k[t]/(t^{2^{d-2}})$.*

The only components of the stable Auslander-Reiten quiver of B containing modules with stable endomorphism ring k are the ones in $(i) - (iii)$.

To prove Proposition 4.2.1 we need the following result which is proved similarly to Lemma 4.1.3.

Lemma 4.2.2. *Let $\Lambda = kQ/I$ with Q and I as in §3.2. For $i \in \{0, 1, 2\}$, let $\mathfrak{C}_i$ be the component of the stable Auslander-Reiten quiver of Λ containing S_i. Let M be an indecomposable Λ-module with $\mathrm{End}_\Lambda(M) = k$. Then M either lies in $\mathfrak{C}_i$ for some i, or M is uniserial of length 2.*

Proof of Proposition 4.2.1. Let $\mathfrak{C}$ be a component of the stable Auslander-Reiten quiver of B containing a module with endomorphism ring k. By Lemma 4.2.2, $\mathfrak{C}$ contains a simple module or a uniserial module of length 2.

Let B_0 be the principal block of $k\mathrm{PSL}_2(\mathbb{F}_q)$ where $q \equiv 1 \mod 4$ and the Sylow 2-subgroups of $\mathrm{PSL}_2(\mathbb{F}_q)$ have order 2^d. By [26], B and B_0 are derived equivalent. By [28, Cor. 5.5], this means that there is a stable equivalence of Morita type between B and B_0 given by a B-B_0-bimodule Ξ. By Lemma 2.2.3 and Remark 2.2.4, if V is a finitely generated B_0-module with stable endomorphism ring k and $V' = \Xi \otimes_{B_0} V$, then V' has stable endomorphism ring k and

$R(\mathrm{PSL}_2(\mathbb{F}_q),V)/2R(\mathrm{PSL}_2(\mathbb{F}_q),V)\cong R(G,V')/2R(G,V')$. Moreover, $V'\cong V''\oplus P$ as kG-modules where P is projective, V'' is indecomposable and $R(G,V'')\cong R(G,V')$.

Because of Remark 2.2.5, to complete the proof of Proposition 4.2.1, we need to find the components of the stable Auslander-Reiten quiver of B_0 and of B, respectively, that correspond to each other under the functor $\Xi\otimes_{B_0}-$, and we need to match up certain modules in these components. Note that by Remark 2.2.5, $\Xi\otimes_{B_0}-$ commutes with the Heller operator Ω. Let $\mathfrak{C}$ be a component of the stable Auslander-Reiten quiver of B containing a module with endomorphism ring k.

Suppose first $\mathfrak{C}$ contains T_0. Then $\mathfrak{C}$ is of type $\mathbb{Z}A_\infty^\infty$. Near T_0, $\mathfrak{C}$ looks as in Figure 4.2.1. Hence $\Omega(\mathfrak{C})$ contains T_1 and T_2. By Proposition 4.1.1, it follows that $\mathfrak{C}$ and $\Omega(\mathfrak{C})$ correspond to

FIGURE 4.2.1. The stable Auslander-Reiten component near T_0.

$$\begin{tikzcd}
\Omega(T_1) \arrow[dr] & & \Omega^{-1}(T_1) \\
& T_0 \arrow[ur] \arrow[dr] & \\
\Omega(T_2) \arrow[ur] & & \Omega^{-1}(T_2)
\end{tikzcd}$$

the components of the stable Auslander-Reiten quiver of B_0 as in Proposition 4.1.1(i). This proves part (i) of Proposition 4.2.1.

Let now $i\neq j$ be in $\{1,2\}$, and suppose $\mathfrak{C}$ contains $T_{0,i}$. Then $\mathfrak{C}$ is a 3-tube and $T_{0,i}$ belongs to its boundary. Since $\Omega^2(T_{j,0})=T_{0,i}$ and $\Omega(T_{0,i})=\Omega^{-2}(T_{0,i})$, $\mathfrak{C}$ also contains $T_{j,0}$ and $\mathfrak{C}=\Omega(\mathfrak{C})$. By Proposition 4.1.1, it follows that for $i=1,2$, the components $\mathfrak{C}$ correspond to the components of the stable Auslander-Reiten quiver of B_0 as in Proposition 4.1.1(ii). This proves part (ii) of Proposition 4.2.1.

Finally, let $i\neq j$ in $\{1,2\}$, and suppose $\mathfrak{C}$ contains $Y=\begin{smallmatrix}T_i\\T_j\end{smallmatrix}$. Then $\mathfrak{C}$ is a component of type $\mathbb{Z}A_\infty^\infty$ and Y is a module of minimal length in $\mathfrak{C}$. Near Y, $\mathfrak{C}$ looks as in Figure 4.2.2. where Y_{1ij}

FIGURE 4.2.2. The stable Auslander-Reiten component near Y.

$$\begin{tikzcd}
\Omega^2(Y_{1ij}) \arrow[dr] & & Y_{1ij} \\
& Y \arrow[ur] \arrow[dr] & \\
Y_{2ij} \arrow[ur] & & \Omega^{-2}(Y_{2ij})
\end{tikzcd}$$

and Y_{2ij} correspond to string modules of $\Lambda=kQ/I$ with Q and I as in §3.2 as follows:

$$Y_{1ij}=\begin{array}{cccc}T_i&&T_0&\\&T_j&&T_i\end{array},\qquad Y_{2ij}=\begin{array}{cccc}T_i&&&\\&T_j&&T_0\\&&T_i&\end{array}.$$

For all $d \geq 3$, Y_{1ij} has a non-trivial endomorphism factoring through T_i which does not factor through a projective B-module, which means that the k-dimension of the stable endomorphism ring of Y_{1ij} is at least 2. If $d = 3$ then $Y_{2ij} = \Omega(Y)$, and so $\Omega(\mathfrak{C}) = \mathfrak{C}$. For $d > 3$, Y_{2ij} has a non-trivial endomorphism factoring through T_i which does not factor through a projective B-module, and thus the k-dimension of the stable endomorphism ring of Y_{2ij} is at least 2. By Proposition 4.1.1 and by Remark 4.1.2, it follows that for $i \neq j$ in $\{1, 2\}$, the components $\mathfrak{C}$ correspond to the components of the stable Auslander-Reiten quiver of B_0 as in Proposition 4.1.1(iii). This implies part (iii) of Proposition 4.2.1.

Since we have matched up all components of the stable Auslander-Reiten quiver of B_0 containing modules with stable endomorphism ring k to certain components of the stable Auslander-Reiten quiver of B, it follows that all other components of the stable Auslander-Reiten quiver of B do not contain any modules with stable endomorphism ring k. This completes the proof of Proposition 4.2.1.

Remark 4.2.3. Similarly to Remark 4.1.7, it follows that if $Y = \begin{matrix} T_i \\ T_j \end{matrix}$ where $i \neq j$ in $\{1, 2\}$, then the universal mod 2 deformation of Y is represented by the uniserial kG-module $\overline{U}$ of length $2 \cdot 2^{d-2}$ with descending radical series

$$(T_i, T_j, T_i, T_j, \dots, T_i, T_j).$$

Moreover the uniserial kG-module $\overline{U'} \cong \overline{U}/Y$ of length $2 \cdot (2^{d-2} - 1)$ defines a lift of Y over $k[t]/(t^{2^{d-2}-1})$.

We next turn to blocks as in §3.3.

Proposition 4.2.4. *Let B be a block of kG which is Morita equivalent to $\Lambda = kQ/I$ with Q and I as in §3.3. Then the order of the defect groups of B is $2^d = 8$. Denote the three simple B-modules by T_0, T_1 and T_2, where T_i corresponds to S_i, for $i \in \{0, 1, 2\}$, under the Morita equivalence, and define Y_{01102} to be the B-module corresponding to the Λ-string-module with underlying string $\beta\alpha^{-1}\gamma\eta$. Let $\mathfrak{C}$ be a component of the stable Auslander-Reiten quiver of B. If $\mathfrak{C}$ contains a module with endomorphism ring k, then $\mathfrak{C}$ contains a simple module or a uniserial module of length 4, or $\Omega(\mathfrak{C})$ contains T_0.*

i. *Suppose $\mathfrak{C}$ contains T_0. Then $\mathfrak{C}$ and $\Omega(\mathfrak{C})$ are both of type $\mathbb{Z}A_\infty^\infty$. All modules M in $\mathfrak{C} \cup \Omega(\mathfrak{C})$ have stable endomorphism ring equal to k and $R(G, M)/2R(G, M) \cong k$.*
ii. *Let $i \neq j$ be in $\{1, 2\}$ and define $T_{i,0,j,0} = \begin{matrix} T_i \\ T_0 \\ T_j \\ T_0 \end{matrix}$ and $T_{0,j,0,i} = \begin{matrix} T_0 \\ T_j \\ T_0 \\ T_i \end{matrix}$. Suppose $\mathfrak{C}$ contains $T_{i,0,j,0}$. Then $\mathfrak{C}$ is a 3-tube with $T_{i,0,j,0}$ and $T_{0,j,0,i}$ belonging to its boundary, and $\mathfrak{C} = \Omega(\mathfrak{C})$. If $i = 2$, $\mathfrak{C}$ also contains T_2. In both cases, if M lies in $\mathfrak{C}$ having stable endomorphism ring k, then $M \in \{T_{i,0,j,0}, \Omega^2(T_{i,0,j,0}), \Omega^4(T_{i,0,j,0})\}$ up to isomorphism, and $R(G, M)/2R(G, M) \cong k$.*
iii. *Suppose $Y \in \{T_1, Y_{01102}\}$ and $\mathfrak{C}$ contains Y. Then $\mathfrak{C} = \Omega(\mathfrak{C})$ is of type $\mathbb{Z}A_\infty^\infty$. If $Y = Y_{01102}$ then $\mathfrak{C}$ contains no B-modules with endomorphism ring k. In both cases, if M lies in $\mathfrak{C}$ having stable endomorphism ring k, then M is isomorphic to $\Omega^j(Y)$ for some integer j, and $R(G, M)/2R(G, M) \cong k[t]/(t^2)$.*

The only components of the stable Auslander-Reiten quiver of B containing modules with stable endomorphism ring k are the ones in $(i) - (iii)$.

The proof of Proposition 4.2.4 is similar to the proof of Proposition 4.2.1 using the following result in place of Lemma 4.2.2.

Lemma 4.2.5. *Let $\Lambda = kQ/I$ with Q and I as in §3.3. For $i \in \{0, 1, 2\}$, let $\mathfrak{C}_i$ be the component of the stable Auslander-Reiten quiver of Λ containing S_i. Let M be an indecomposable Λ-module*

with $\mathrm{End}_\Lambda(M) = k$*. Then* M *either lies in* $\mathfrak{C}_0 \cup \Omega(\mathfrak{C}_0)$*, or* M *lies in* $\mathfrak{C}_i$ *for an* $i \in \{1,2\}$*, or* M *is uniserial of length* 4*.*

Remark 4.2.6. Similarly to Remark 4.1.7, it follows that the universal mod 2 deformation of T_1 is represented by the uniserial kG-module $\overline{U}_1$ with radical series (T_1, T_1).

The universal mod 2 deformation of Y_{01102} can be obtained as follows. There is a non-split short exact sequence of kG-modules

$$0 \to \Omega(Y_{01102}) \to P_{T_1} \oplus P_{T_2} \to Y_{01102} \to 0$$

where P_{T_1} (resp. P_{T_2}) is the projective indecomposable kG-module with top T_1 (resp. T_2). Moreover, there is a natural surjection $\Omega(Y_{01102}) \to Y_{01102}$ with kernel isomorphic to T_2. Thus there is a non-split short exact sequence of kG-modules

$$0 \to Y_{01102} \to P_{T_1} \oplus P_{T_2}/T_2 \to Y_{01102} \to 0 \tag{4.2.1}$$

representing a non-zero element in $\mathrm{Ext}^1_{kG}(Y_{01102}, Y_{01102})$. Hence the universal mod 2 deformation of Y_{01102} is represented by the kG-module $P_{T_1} \oplus P_{T_2}/T_2$.

5. Universal deformation rings

We assume the notations from §3 and §4. In this section, we determine the universal deformation rings of all finitely generated kG-modules which belong to B and have stable endomorphism ring equal to k. Since the case $d = 2$ has been done in [4], we assume throughout this section that $d \geq 3$.

In §5.1, we state our main Theorem 5.1 and some consequences concerning local complete intersections (see Corollary 5.1.2). In §5.2, we analyze the modules belonging to the boundaries of 3-tubes. In §5.3, we then use these results together with the results from §3.4 to prove Theorem 5.1.

5.1. **The Main Theorem.**

Theorem 5.1. *Let* G *be a finite group, and let* B *be a block of* kG *with dihedral defect group* D *of order* 2^d *where* $d \geq 3$*. Assume that* B *is Morita equivalent to the principal* 2*-modular block of a finite simple group. Then* B *is Morita equivalent to a block as in §3.1, §3.2 or §3.3. Let* V *be a finitely generated indecomposable* B*-module with stable endomorphism ring* k*, and let* $\mathfrak{C}$ *be the component of the stable Auslander-Reiten quiver of* B *containing* V*.*

i. *If* $\mathfrak{C}$ *or* $\Omega(\mathfrak{C})$ *is as in part* (i) *of Propositions 4.1.1, 4.2.1 or 4.2.4, then* $R(G,V)$ *is isomorphic to a quotient ring of* W*.*
ii. *If* $\mathfrak{C}$ *or* $\Omega(\mathfrak{C})$ *is as in part* (ii) *of Propositions 4.1.1, 4.2.1 or 4.2.4, then* $R(G,V)$ *is isomorphic to* k*.*
iii. *If* $\mathfrak{C}$ *or* $\Omega(\mathfrak{C})$ *is as in part* (iii) *of Propositions 4.1.1, 4.2.1 or 4.2.4, then* $R(G,V)$ *is isomorphic to*

$$W[[t]]/(p_d(t)(t-2), 2\,p_d(t))$$

as a W*-algebra where* $p_d(t) \in W[t]$ *is as in Definition 2.3.4.*

In all cases $(i) - (iii)$*,* $R(G,V)$ *is isomorphic to a subquotient ring of* WD*. The only components of the stable Auslander-Reiten quiver of* B *containing modules with stable endomorphism ring* k *are the ones in* $(i) - (iii)$*.*

Remark 5.1.1. Using Lemma 2.3.2, we can prove that if V belongs to $\mathfrak{C}$ as in part (i) of Theorem 5.1 then $R(G,V) \cong W$.

Corollary 5.1.2. *Assuming the notations of Theorem 5.1, suppose* $\mathfrak{C}$ *is as in part* (iii) *of Propositions 4.1.1, 4.2.1 or 4.2.4. Let* V *be a finitely generated indecomposable* kG*-module in* $\mathfrak{C}$ *with stable endomorphism ring* k*. Then* $R(G,V)$ *is not a complete intersection ring.*

In particular, there is an infinite series of finite groups G *and indecomposable* kG*-modules* V *such that* $R(G,V)$ *is not a complete intersection. This series is given by* $G = \mathrm{PSL}_2(\mathbb{F}_q)$ *for* q *an odd prime power and* 8 *dividing* $\#G$*, and* $V = \Omega^i(V')$ *for* $i \in \mathbb{Z}$*, where* V' *is a uniserial* kG*-module*

belonging to the principal block of kG with radical series length 4 (*resp. radical series length* 2 *and no composition factor isomorphic to the trivial simple module*) *in case* $q \equiv 1 \mod 4$ (*resp.* $q \equiv 3 \mod 4$). *Moreover, in case* $G = A_7$, *the unique* (*up to isomorphism*) *irreducible kG-module V of dimension* 14 *provides an example of an irreducible V such that $R(G, V)$ is not a complete intersection.*

5.2. **Modules at the boundaries of three-tubes.** We first summarize the known facts about the modules belonging to the boundaries of 3-tubes in the stable Auslander-Reiten quiver of one of the blocks B under consideration. These facts can be found e.g. in [20, Chapter V].

Facts 5.2.1. Let G be a finite group, and let B be a block of kG with dihedral defect group D of order 2^d where $d \geq 3$. Assume that B contains exactly three isomorphism classes of simple kG-modules. Then the stable Auslander-Reiten quiver of B has exactly two 3-tubes. Suppose V is a finitely generated indecomposable B-module belonging to the boundary of a 3-tube. Then the vertices of V are Klein four groups. Let K be a vertex of V.

i. The quotient group $N_G(K)/C_G(K)$ is isomorphic to a symmetric group S_3.
ii. There is a block b of $kN_G(K)$ with $b^G = B$ such that the Green correspondent fV of V belongs to the boundary of a 3-tube in the stable Auslander-Reiten quiver of b. Moreover, b is Morita equivalent to kS_4 modulo the socle.

Lemma 5.2.2. *Suppose K and b are as in Facts 5.2.1. Let $H = N_G(K)$, let $C = C_G(K)$, and let N be a normal subgroup of H of index* 2 *containing C.*

i. *There is a unique block b_1 of kN with defect group K which is covered by b. Moreover, $b_1^H = b$ and b_1 is Morita equivalent to kA_4.*
ii. *Let $g \in H$ with $\langle gN \rangle = H/N$. Then there is a simple b_1-module S_0 with $g(S_0) \cong S_0$, where $g(S_0)$ denotes the kN-module such that gxg^{-1} acts on $g(S_0)$ the same way as $x \in N$ acts on S_0. The other two representatives of non-isomorphic simple b_1-modules S_1 and S_2 satisfy $g(S_1) \cong S_2$ and $g(S_2) \cong S_1$.*
iii. *The stable Auslander-Reiten quiver of b_1 has two* 3*-tubes whose modules all have vertex K. Each b_1-module at the end of one* 3*-tube has as source the band module $S_\omega = M(XY^{-1}, \omega, 1)$, and each b_1-module at the end of the other* 3*-tube has as source the band module $S_{\omega^2} = M(XY^{-1}, \omega^2, 1)$, where XY^{-1} denotes the single band of kK and ω is a primitive cube root of unity in k.*

Proof. It follows from [1, Thm. 15.1] that there is a unique block b_1 of kN with defect group K which is covered by b, and that $b_1^H = b$. Since $N/C_N(K)$ has order 3, it follows e.g. from [20, Proof of Prop. V.2.14] that b_1 is Morita equivalent to kA_4. This implies part (i).

Now we turn to part (ii). Since there are three isomorphism classes of simple b_1-modules and $[H : N] = 2$, there is a simple b_1-module S_0 with $g(S_0) \cong S_0$. So we only need to show that $g(S_1)$ is not isomorphic to S_1. To get a contradiciton, suppose that $g(S_1) \cong S_1$, and hence also $g(S_2) \cong S_2$. For $i = 0, 1, 2$, consider the kH-module $X_i = \mathrm{Ind}_N^H(S_i)$. Then by Mackey's Theorem $(X_i)_N \cong S_i \oplus g(S_i) \cong S_i \oplus S_i$. In particular, X_i is a b-module. If Y_i is a simple b-module in the socle of X_i then $(Y_i)_N$ is a submodule of $(X_i)_N \cong S_i \oplus S_i$. Since S_0, S_1 and S_2 are pairwise non-isomorphic, it follows that Y_0, Y_1 and Y_2 are pairwise non-isomorphic. But this is a contradiction to b having only two non-isomorphic simple modules. Hence $g(S_1) \cong S_2$ and vice versa.

Finally, we turn to part (iii). It follows e.g. from [20, Proof of Thm. V.4.1] that all the b_1-modules in a 3-tube have the same vertex and that this vertex must be a Klein four group. Since K is a defect group of b_1, K is a vertex. Let $\tilde{\nu} \in N$ such that $\langle \tilde{\nu} C \rangle = N/C$. Then $\tilde{\nu}$ acts on K by conjugation which induces an automorphism ν of K of order 3. Let $A = kK * \langle \nu \rangle$ be the skew group ring of kK with $\langle \nu \rangle$. Then $A \cong kA_4$ (see e.g. [20, Cor. V.2.4.1]). Let β be a block of kC which is covered by b_1. Then by [20, Proof of Prop. V.2.14], β is Morita equivalent to kK. Let P be the unique projective indecomposable β-module (up to isomorphism) and identify P with an inner direct summand of kC. Then $\mathrm{End}_{kC}(P) = kK$. Since K is central in C and we assume $P \subseteq kC$, right multiplication by any element of kK defines a kC-module endomorphism of P. If

$P^N = \mathrm{Ind}_C^N(P)$ then $\mathrm{End}_{kN}(P^N) \cong A = kK * \langle \nu \rangle$ by [20, Proof of Prop. V.2.14]. Moreover, it follows from [20, §V.2.9] and [20, Proof of Prop. V.2.14] that the right action of kK on P by right multiplication extends to a right action of A on P^N. In particular, the elements of kK act by right multiplication on P^N when identifying P^N with an inner direct summand of kN. In [15], the indecomposable A-modules are described as direct summands of induced indecomposable kK-modules. This shows that the direct sum of the A-modules belonging to the boundary of a 3-tube of the stable Auslander-Reiten quiver of A is isomorphic to $A \otimes_{kK} S_\rho$ where for one 3-tube $\rho = \omega$ and for the other 3-tube $\rho = \omega^2$ and S_ρ is as in part (iii) of the statement of the lemma. Since P^N gives a Morita equivalence between b_1 and A, the functor $P^N \otimes_A -$ sends the A-modules belonging to the boundary of a 3-tube of the stable Auslander-Reiten quiver of A to b_1-modules which belong to the boundary of a 3-tube of the stable Auslander-Reiten quiver of b_1. Thus the direct sum of the b_1-modules belonging to the boundary of a 3-tube is isomorphic to

$$P^N \otimes_A (A \otimes_{kK} S_\rho) \cong P^N \otimes_{kK} S_\rho$$

where kK acts on P^N by right multiplication and $\rho = \omega$ or ω^2. Since we view P^N as an inner direct summand of kN, it follows that the direct sum of the b_1-modules belonging to the boundary of a 3-tube is isomorphic to a direct summand of $kN \otimes_{kK} S_\rho = \mathrm{Ind}_K^N S_\rho$. But this means that S_ρ is a source of each of the b_1-modules belonging to the boundary of the 3-tube under consideration. This completes the proof of Lemma 5.2.2. □

Lemma 5.2.3. *Assume the notations from Lemma 5.2.2. Let E_0 and E be the two non-isomorphic simple b-modules such that $U_E = \begin{matrix} E \\ E \end{matrix}$ belongs to the boundary of the 3-tube of the stable Auslander-Reiten quiver of b. Then U_E is isomorphic to the induced module $\mathrm{Ind}_N^H M$ for some b_1-module M which belongs to the boundary of a 3-tube of the stable Auslander-Reiten quiver of b_1. Moreover, K is a vertex of U_E, and the source of U_E is $S_E = S_\omega$ which is conjugate to S_{ω^2} in $N_H(K) = H$. In particular, the restriction of S_E to any proper subgroup of K is projective.*

Proof. For $i \in \{0, 1, 2\}$, let $X_i = \mathrm{Ind}_N^H S_i$. By Mackey's Theorem, $(X_i)_N \cong S_i \oplus g(S_i)$, where g is as in part (ii) of Lemma 5.2.2. Hence $(X_0)_N \cong S_0 \oplus S_0$ and $(X_i)_N \cong S_1 \oplus S_2$ for $i = 1, 2$. In particular, for $i \in \{0, 1, 2\}$, X_i is a b-module. Let Y_i be a simple b-module in the socle of X_i. Since $g(Y_i) \cong Y_i$, it follows that $X_1 \cong X_2$ is simple. Call this simple b-module Y_{12}. By [1, Lemmas 8.5 and 8.6], induced modules of projective modules are projective, and the induced modules of the composition factors of a b_1-module T provide the composition factors of the induced module $\mathrm{Ind}_N^H T$. Since the projective cover of S_0 has composition factors S_0, S_1, S_2, S_0 and since the projective covers of E_0 and E have 6, resp. 5, composition factors, it follows that $\mathrm{Ind}_N^H S_0$ has composition factors Y_0 and Y_0. This implies that $Y_0 = E_0$ and $Y_{12} = E$.

Now consider the indecomposable b_1-module $M = \begin{matrix} S_1 \\ S_2 \end{matrix}$, which belongs to the boundary of a 3-tube of the stable Auslander-Reiten quiver of b_1. By [1, Lemma 8.6(5)], it follows that $\mathrm{Ind}_N^H M$ satisfies a non-split short exact sequence

$$0 \to E \to \mathrm{Ind}_N^H M \to E \to 0,$$

which implies $U_E = \begin{matrix} E \\ E \end{matrix} \cong \mathrm{Ind}_N^H M$.

It follows e.g. from [20, Proof of Thm. V.4.1] that all the modules in 3-tubes of the stable Auslander-Reiten quiver of b have Klein four groups as vertices. By Lemma 5.2.2(iii), S_ρ is a source of M where $\rho = \omega$ or ω^2. Hence U_E is a direct summand of $\mathrm{Ind}_N^H(\mathrm{Ind}_K^N S_\rho) \cong \mathrm{Ind}_K^H S_\rho$, which means that K is a vertex of U_E and S_ρ is a source of U_E. But similarly, we also get that S_{ρ^2} is a source of U_E. Because sources are unique up to conjugation in $N_H(K) = H$, S_ω is conjugate to S_{ω^2} in H. Since $S_E = S_\omega$ is 2-dimensional over k and each proper subgroup of K of order 2 acts non-trivially on S_E, the restriction of S_E to any proper subgroup of K is projective. This completes the proof of Lemma 5.2.3. □

Proposition 5.2.4. *Assume the notations of Lemmas 5.2.2 and 5.2.3. If U is an indecomposable b-module belonging to the boundary of the 3-tube of the stable Auslander-Reiten quiver of b, then U has a universal deformation ring with $R(H,U) \cong k$.*

Proof. It follows that U lies in the Ω-orbit of the indecomposable b-module U_E. By Lemma 5.2.3, $U_E \cong \mathrm{Ind}_N^H M$ for some b_1-module M which belongs to the boundary of a 3-tube of the stable Auslander-Reiten quiver of b_1. Since b_1 is Morita equivalent to kA_4, it follows from [4, Prop. 3.4] that $\underline{\mathrm{End}}_{kN}(M) = k$ and $R(N,M) \cong k$. Since $\underline{\mathrm{End}}_{kH}(U_E) = k$ and

$$\dim_k \mathrm{Ext}^1_{kH}(U_E,U_E) = 0 = \dim_k \mathrm{Ext}^1_{kN}(M,M),$$

Proposition 2.1.3 implies that $R(H,U_E) \cong R(N,M) \cong k$. By Lemma 2.1.2, this implies Proposition 5.2.4. □

Corollary 5.2.5. *Assume the notations of Theorem 5.1 and suppose $\mathfrak{C}$ is as in part (ii) of Theorem 5.1. Let V be a finitely generated kG-module in $\mathfrak{C}$ with stable endomorphism ring k. Then $R(G,V) \cong k$.*

Proof. It follows from part (ii) of Propositions 4.1.1, 4.2.1 or 4.2.4 that V is a finitely generated indecomposable B-module belonging to the boundary of a 3-tube. Hence we may use the notations of Facts 5.2.1, Lemmas 5.2.2, 5.2.3 and Proposition 5.2.4. We claim that

$$\mathrm{Ind}_H^G fV \cong V \oplus \text{ projectives.} \tag{5.2.1}$$

Since fV belongs to the boundary of the 3-tube of the stable Auslander-Reiten quiver of b, it lies in the Ω-orbit of U_E from Lemma 5.2.3. Since the Green correspondence commutes with Ω (see e.g. [1, Prop. 20.7]), it is enough to show (5.2.1) in case $fV = U_E$. Using Green correspondence, we know $\mathrm{Ind}_H^G U_E \cong V \oplus X$ where X is relatively $\mathfrak{x}$-projective and the groups in $\mathfrak{x}$ have the form $sKs^{-1} \cap K$ for some $s \in G$, $s \notin H$. Suppose there is an indecomposable summand Y of X which has a non-trivial vertex Q. Then Q has order 2. Because $\mathrm{Ind}_H^G U_E$ is a direct summand of $\mathrm{Ind}_K^G S_E$, we get that Y_Q is a direct summand of

$$(\mathrm{Ind}_K^G S_E)_Q \cong \bigoplus_{t \in Q\backslash G/K} \mathrm{Ind}^Q_{Q\cap tKt^{-1}}(t(S_E)_{Q\cap tKt^{-1}}).$$

Since each term $t(S_E)_{Q\cap tKt^{-1}}$ is projective by Lemma 5.2.3, it follows that Y_Q is projective. But Y is a direct summand of $\mathrm{Ind}_Q^G(Y_Q)$, hence Y is projective. This is a contradiction to Q being a non-trivial group. This implies (5.2.1). By Proposition 2.1.3 it follows that $R(G,V) \cong R(H,fV)$. Hence Corollary 5.2.5 follows from Proposition 5.2.4. □

5.3. **Proof of Theorem 5.1.** Part (i) and part (ii) of Theorem 5.1 and the very last statement about the components of the stable Auslander-Reiten quiver containing modules with stable endomorphism ring k follow from Propositions 4.1.1, 4.2.1 or 4.2.4 together with Corollary 5.2.5. Since it follows from Lemma 2.3.6 that $W[[t]]/(p_d(t)(t-2), 2\,p_d(t))$ is isomorphic to a subquotient ring of WD when $p_d(t) \in W[t]$ is as in Definition 2.3.4, it only remains to prove part (iii) of Theorem 5.1.

Let $\mathfrak{C}$ be as in part (iii) of Theorem 5.1, and let V be a finitely generated indecomposable kG-module with stable endomorphism ring k belonging to $\mathfrak{C}$. By part (iii) of Propositions 4.1.1, 4.2.1 or 4.2.4, $R(G,V)/2R(G,V) \cong k[t]/(t^{2^{d-2}})$.

We first look at the case when B is as in §3.1. By Proposition 4.1.1(iii) and Remark 4.1.2, it is enough to consider V where V is either $Y_1 = \begin{smallmatrix} T_1 \\ T_0 \\ T_2 \\ T_0 \end{smallmatrix}$ or $Y_2 = \begin{smallmatrix} T_2 \\ T_0 \\ T_1 \\ T_0 \end{smallmatrix}$. Let $V = Y_1$. (The case $V = Y_2$ is proved similarly.) By Remark 4.1.7, the universal mod 2 deformation of Y_1 is represented by the uniserial kG-module $\overline{U}$ with descending radical series

$$(T_1, T_0, T_2, T_0, T_1, T_0, T_2, T_0, \ldots, T_1, T_0, T_2, T_0)$$

of length $4\cdot 2^{d-2}$. Moreover, the uniserial kG-module $\overline{U'}=\overline{U}/Y_1$ of length $4\cdot(2^{d-2}-1)$ defines a lift of Y_1 over $k[t]/(t^{2^{d-2}-1})$. We show that there is a surjective continuous W-algebra homomorphism $\tau: R\to R'$ where $R'=W[[t]]/(p_d(t))$ is as in Definition 2.3.4. We use the notations from §3.4. It follows from the decomposition matrix in Figure 3.1.3 and [17, Prop. (23.7)] that there is a W-pure WG-sublattice X' of the projective indecomposable WG-module P_1^W with top T_1 such that $U'=P_1^W/X'$ has F-character

$$\sum_{\ell=2}^{d-1}\rho_\ell=\sum_{i=1}^{2^{d-2}-1}\chi_{5,i}.$$

Then $U'/2U'$ is an indecomposable kG-module with top T_1 which has the same decomposition factors as $\overline{U'}$ and is thus isomorphic to $\overline{U'}$. Hence U' is a lift of $\overline{U'}$ over W. We want to show that U' is in fact a lift of Y_1 over R'. We first prove that R' is isomorphic to a W-subalgebra of $\mathrm{End}_{WG}(U')$. Let σ be an element of order 2^{d-1} in D, and let $t(C)$ be the class sum of the conjugacy class C of σ in G. Since $t(C)$ lies in the center of WG, multiplication by $t(C)$ defines a WG-module endomorphism of U'. Since $\mathrm{End}_{WG}(U')$ can naturally be identified with a subring of $\mathrm{End}_{FG}(F\otimes_W U')$, $t(C)$ acts on U' as multiplication by a scalar in $F\otimes_W R'$. This scalar can be read off from the action of $t(C)$ on $F\otimes_W U'=\bigoplus_{\ell=2}^{d-1}V_\ell$. By (3.4.5), there exists a unit ω in W such that for $2\le\ell\le d-1$, the action of $t(C)$ on V_ℓ is given as multiplication by

$$\omega\cdot(\zeta_{2^{d-1}}^{2^{d-1-\ell}}+\zeta_{2^{d-1}}^{-2^{d-1-\ell}})$$

when we identify $\mathrm{End}_{FG}(V_\ell)$ with $\mathrm{End}_{F(\zeta_{2^\ell}+\zeta_{2^\ell}^{-1})G}(X_\ell)$ for an absolutely irreducible $F(\zeta_{2^\ell}+\zeta_{2^\ell}^{-1})G$-constituent X_ℓ of V_ℓ with character $\chi_{5,2^{d-1-\ell}}$. Recall from Remark 2.3.5 that R' can be identified with the W-subalgebra of $\bigoplus_{\ell=2}^{d-1}W[\zeta_{2^\ell}+\zeta_{2^\ell}^{-1}]$ generated by $(\zeta_{2^\ell}^{r_\ell}+\zeta_{2^\ell}^{-r_\ell})_{\ell=2}^{d-1}$ for any sequence $(r_\ell)_{\ell=2}^{d-1}$ of odd numbers. This implies that R' is isomorphic to a W-subalgebra of $\mathrm{End}_{WG}(U')$, and hence U' is an $R'G$-module. We next prove that U' is free as an R'-module. Since U' is finitely generated as a W-module, it is also finitely generated as an R'-module. Since R' is a local ring, it follows by Nakayama's Lemma that any k-basis $\{\overline{b}_1,\dots,\overline{b}_s\}$ of $U'/\max(R')U'\cong Y_1$ can be lifted to a set $\{b_1,\dots,b_s\}$ of generators of U' over R'. Since $F\otimes_W U'$ is a free $(F\otimes_W R')$-module of rank s, it follows that $b_1,\dots,b_s$ are linearly independent over R'. Since $\mathrm{End}_{FG}(F\otimes_W U')\cong F\otimes_W R'$, this then implies that $\mathrm{End}_{WG}(U')\cong R'$. Hence U' is a lift of Y_1 over R'. We therefore have a continuous W-algebra homomorphism $\tau:R\to R'$ relative to U'. Since $U'/2U'$ is indecomposable as a kG-module, τ must be surjective. By Lemma 2.3.3, it follows that $R(G,V)\cong W[[t]]/(p_d(t)(t-2c),a2^mp_d(t))$ for some $c\in W$, $a\in\{0,1\}$ and $0<m\in\mathbb{Z}$. If $a=0$ (resp. $a=1$), the natural projection $R(G,V)\to W[[t]]/(p_d(t)(t-2c))$ (resp. $R(G,V)\to(W/2^mW)[[t]]/(p_d(t)(t-2c))$) gives a lift of $\overline{U}$, when regarded as a kG-module, over W (resp. over $W/2^mW$). But $\overline{U}\cong\Omega^{-1}(T_1)$, which implies $R(G,\overline{U})\cong k$ by Corollary 5.2.5. Hence $a=1$ and $m=1$, and part (iii) of Theorem 5.1 follows in case B is as in §3.1.

The case when B is as in §3.2 is proved similarly to the case when B is as in §3.1, using Proposition 4.2.1(iii), Remark 4.2.3 and the decomposition matrix in Figure 3.2.3.

Finally we look at the case when B is as in §3.3. By Proposition 4.2.4(iii), it is enough to consider $V\in\{T_1,Y_{01102}\}$. If we show that V has a lift over W, then it follows by Lemma 2.3.3, using Remark 4.2.6 and Corollary 5.2.5, that $R(G,V)\cong W[[t]]/(t(t-2),2\,t)$. Using the decomposition matrix in Figure 3.3.3 and [17, Prop. (23.7)] we see that there is a lift of T_1 over W. To see that Y_{01102} has a lift over W, we consider the two uniserial kG-modules $Z_1=\begin{matrix}T_1\\T_0\end{matrix}$ and $Z_2=\begin{matrix}T_2\\T_0\\T_1\end{matrix}$. Then there is a non-split short exact sequence of kG-modules

$$0\to Z_2\to Y_{01102}\to Z_1\to 0$$

with $\mathrm{Ext}^1_{kG}(Z_1,Z_2)=k$. It follows from the decomposition matrix in Figure 3.3.3 and [17, Prop. (23.7)] that there exists a lift X_i of Z_i over W for $i=1,2$. Moreover, the F-character of $F\otimes_W X_1$

(resp. of $F \otimes_W X_2$) is χ_2 (resp. χ_4). In particular, $\mathrm{Hom}_{FG}(F \otimes_W X_1, F \otimes_W X_2) = 0$. Since $\mathrm{Hom}_{kG}(Z_1, Z_2) = k$, it follows from Lemma 2.3.2 that there is a lift of Y_{01102} over W. This completes the proof of Theorem 5.1.

6. Stable endomorphism rings and Ext groups

Assume the notations of §4. In this section we complete the proof of Proposition 4.1.1, by determining which components of the stable Auslander-Reiten quiver of a block B as in §3.1 contain modules with stable endomorphism ring k and by determining the Ext groups for these modules. Since there are stable equivalences of Morita type between B and blocks as in §3.2 or §3.3, we determine at the same time which components of the stable Auslander-Reiten quiver of the blocks in §3.2 and in §3.3 contain modules with stable endomorphism ring k and also determine the Ext groups for these modules.

Recall that B is Morita equivalent to the basic algebra Λ of a special biserial algebra, so we can use the description of indecomposable Λ-modules as string and band modules (see §7). In §6.1, we give a description of the homomorphisms between string modules as determined in [25] and provide a short-hand notation for such homomorphisms. We also give a criterion which helps determine stable homomorphisms between string modules. In §6.2, we prove Lemmas 4.1.4, 4.1.5 and 4.1.6. In §6.3, we consider all components of the stable Auslander-Reiten quiver of type $\mathbb{Z}A_\infty^\infty$ and prove that the only such components containing a module with stable endomorphism ring k are precisely those components containing a module M with $\mathrm{End}_\Lambda(M) = k$ or $\mathrm{End}_\Lambda(\Omega(M)) = k$. In §6.4, we consider the components of the stable Auslander-Reiten quiver which are 1-tubes and prove that no 1-tube contains any modules with stable endomorphism ring k.

We freely use §7 without always explicitly referring to particular results. We especially use the phrase "canonical k-basis" for string modules as introduced in §7.1.1 to be able to readily write down homomorphisms between string modules.

6.1. **Homomorphisms between string modules.**

Remark 6.1.1. Let $\Lambda = kQ/I$ be a basic special biserial algebra, and let $M(S)$ (resp. $M(T)$) be a string module with canonical k-basis $\{x_u\}_{u=0}^m$ (resp. $\{y_v\}_{v=0}^n$) relative to the representative S (resp. T). Suppose C is a string such that

i. $S \sim_s S_1 C S_2$ with (S_1 of length 0 or $S_1 = S_1'\zeta_1$) and (S_2 of length 0 or $S_2 = \zeta_2^{-1}S_2'$), where S_1, S_1', S_2, S_2' are strings and ζ_1, ζ_2 are arrows in Q; and
ii. $T \sim_s T_1 C T_2$ with (T_1 of length 0 or $T_1 = T_1'\xi_1^{-1}$) and (T_2 of length 0 or $T_2 = \xi_2 T_2'$), where T_1, T_1', T_2, T_2' are strings and ξ_1, ξ_2 are arrows in Q.

Then there exists a Λ-module homomorphism $\sigma_C : M(S) \to M(T)$ which factors through $M(C)$ and which sends each element of $\{x_u\}_{u=0}^m$ either to zero or to an element of $\{y_v\}_{v=0}^n$, according to the relative position of C in S and T, respectively. If e.g. $S = s_1 s_2 \cdots s_m$, $T = t_1 t_2 \cdots t_n$, and $C = s_{i+1}s_{i+2}\cdots s_{i+\ell} = t_{j+\ell}^{-1}t_{j+\ell-1}^{-1}\cdots t_{j+1}^{-1}$, then

$$\sigma_C(x_{i+t}) = y_{j+\ell-t} \text{ for } 0 \le t \le \ell, \text{ and } \sigma_C(x_u) = 0 \text{ for all other } u.$$

Note that there may be several choices of S_1, S_2 (resp. T_1, T_2) in (i) (resp. (ii)). In other words, there may be more than one homomorphism factoring through $M(C)$. By [25], every Λ-module homomorphism $\sigma : M(S) \to M(T)$ is a k-linear combination of homomorphisms which factor through string modules corresponding to strings C satisfying (i) and (ii).

The following definition provides a short-hand notation for homomorphisms between string modules, relative to fixed choices of canonical k-bases.

Definition 6.1.2. Let $\Lambda = kQ/I$ be a basic special biserial algebra. Suppose $X = M(S)$ (resp. $Y = M(T)$) is a string module with canonical k-basis $\{x_u\}_{u=0}^m$ (resp. $\{y_v\}_{v=0}^n$) relative to the representative S (resp. T).

i. Suppose there exist $0 \le i \le m$, $0 \le j \le n$ and $0 \le \ell \le \min\{m-i, j\}$ such that $\alpha : X \to Y$ defined by
$$\alpha(x_{i+t}) = y_{j-\ell+t} \text{ for } 0 \le t \le \ell, \text{ and } \alpha(x_u) = 0 \text{ for all other } u$$
is a Λ-module homomorphism. Then we denote α by $\mathrm{hom}^{++}_{(X,Y)}(x_i, y_j, \ell)$.

ii. Suppose there exist $0 \le i \le m$, $0 \le j \le n$ and $0 \le \ell \le \min\{m-i, n-j\}$ such that $\beta : X \to Y$ defined by
$$\beta(x_{i+t}) = y_{j+\ell-t} \text{ for } 0 \le t \le \ell, \text{ and } \beta(x_u) = 0 \text{ for all other } u$$
is a Λ-module homomorphism. Then we denote β by $\mathrm{hom}^{+-}_{(X,Y)}(x_i, y_j, \ell)$.

iii. Suppose there exist $0 \le i \le m$, $0 \le j \le n$ and $0 \le \ell \le \min\{i, j\}$ such that $\gamma : X \to Y$ defined by
$$\gamma(x_{i-t}) = y_{j-\ell+t} \text{ for } 0 \le t \le \ell, \text{ and } \gamma(x_u) = 0 \text{ for all other } u$$
is a Λ-module homomorphism. Then we denote γ by $\mathrm{hom}^{-+}_{(X,Y)}(x_i, y_j, \ell)$.

iv. Suppose there exist $0 \le i \le m$, $0 \le j \le n$ and $0 \le \ell \le \min\{i, n-j\}$ such that $\delta : X \to Y$ defined by
$$\delta(x_{i-t}) = y_{j+\ell-t} \text{ for } 0 \le t \le \ell, \text{ and } \delta(x_u) = 0 \text{ for all other } u$$
is a Λ-module homomorphism. Then we denote δ by $\mathrm{hom}^{--}_{(X,Y)}(x_i, y_j, \ell)$.

If $X = Y$ then we write end_X instead of $\mathrm{hom}_{(X,Y)}$.

Note that in many of our applications of Definition 6.1.2, x_i corresponds to a source in the linear quiver Q_S defining $M(S)$ and y_j corresponds to a sink in the linear quiver Q_T defining $M(T)$.

We now illustrate cases (i) and (ii) of Definition 6.1.2. Suppose the string S (resp. T) is the word $S = s_1 s_2 \cdots s_m$ (resp. $T = t_1 t_2 \cdots t_n$). Then $\alpha = \mathrm{hom}^{++}_{(X,Y)}(x_i, y_j, \ell)$ factors as
$$\alpha: \quad X = M(S) \xrightarrow{\pi} M(s_{i+1}s_{i+2}\cdots s_{i+\ell}) \xrightarrow{\cong} M(t_{j-\ell+1}t_{j-\ell+2}\cdots t_j) \xrightarrow{\iota} M(T) = Y$$
where $s_{i+1}s_{i+2}\cdots s_{i+\ell} = t_{j-\ell+1}t_{j-\ell+2}\cdots t_j$ as words, and π is the canonical projection and ι the canonical injection. On the other hand, $\beta = \mathrm{hom}^{+-}_{(X,Y)}(x_i, y_j, \ell)$ factors as
$$\beta: \quad X = M(S) \xrightarrow{\pi} M(s_{i+1}s_{i+2}\cdots s_{i+\ell}) \xrightarrow{\cong} M(t^{-1}_{j+\ell}t^{-1}_{j+\ell-1}\cdots t^{-1}_{j+1}) \xrightarrow{\iota} M(T) = Y$$
where $s_{i+1}s_{i+2}\cdots s_{i+\ell} = t^{-1}_{j+\ell}t^{-1}_{j+\ell-1}\cdots t^{-1}_{j+1}$ as words.

The following is an easy combinatorial Lemma which helps determine stable homomorphisms.

Lemma 6.1.3. *Let $\Lambda = kQ/I$ be a basic special biserial algebra. Suppose $0 < \mu \in \mathbb{Z}$ and $0 \le a < \mu$. Let $M(S)$ and $M(T)$ be string modules with canonical k-bases $\{x_u\}$ and $\{y_v\}$ relative to the representatives S and T, respectively. Suppose that $\{h_i\}_{i=1}^s$ (resp. $\{f_j\}_{j=1}^t$) are subsets of $\{x_u\}$ (resp. $\{y_v\}$). Let $\epsilon \in \{\pm 1, 0\}$ be the sign of $(t-s)$. For $1 \le i \le s$ and $1 \le j \le t$ with $j - i \equiv a \mod \mu$, assume that the map*
$$\lambda_{i,j} : M(S) \to M(T) \text{ defined by } \lambda_{i,j}(h_i) = f_j \text{ and } \lambda_{i,j}(x_u) = 0 \text{ for } x_u \ne h_i$$
is a Λ-module homomorphism. Suppose that for all $1 \le i \le s-1$ and $1 \le j \le t-1$ with $j - i \equiv a \mod \mu$, $\alpha_{i,j} = \lambda_{i,j} + \lambda_{i+1,j+1}$ factors through a projective Λ-module. Suppose further that either

i. *$\epsilon \ge 0$, and each $\lambda_{1,j}$ $(1 + \delta_1 \le j \le t,\ j - 1 \equiv a \mod \mu)$ and each $\lambda_{s,j}$ $(1 \le j \le t - \delta_2,\ j - s \equiv a \mod \mu)$ factors through a projective Λ-module for $\{\delta_1, \delta_2\} = \{1, \epsilon\}$; or*

ii. *$\epsilon \le 0$, and each $\lambda_{i,1}$ $(1 - \delta_1 \le i \le s,\ 1 - i \equiv a \mod \mu)$ and each $\lambda_{i,t}$ $(1 \le i \le s + \delta_2,\ t - i \equiv a \mod \mu)$ factors through a projective Λ-module for $\{\delta_1, \delta_2\} = \{-1, \epsilon\}$.*

Then for all $1 \le i \le s$ and $1 \le j \le t$ with $j - i \equiv a \mod \mu$, $\lambda_{i,j}$ factors through a projective Λ-module.

6.2. **Components containing modules with endomorphism ring k.** In this subsection, we prove Lemmas 4.1.4, 4.1.5 and 4.1.6. Let $\Lambda = kQ/I$ with Q and I as in §3.1.

Proof of Lemma 4.1.4. Let $\mathcal{C}_0$ be the component of the stable Auslander-Reiten quiver of Λ containing S_0, and let M be a Λ-module belonging to $\mathcal{C}_0 \cup \Omega(\mathcal{C}_0)$. We need to show that $\underline{\mathrm{End}}_\Lambda(M) = k$ and $\mathrm{Ext}^1_\Lambda(M, M) = 0$.

Using the description of the components of the stable Auslander-Reiten quiver of Λ as in §7.2, we see that $\mathfrak{C}_0$ is of type $\mathbb{Z}A^\infty_\infty$. Moreover, using hooks and cohooks (see §7.2) we obtain the following. There is an $i \in \mathbb{Z}$ such that $\Omega^i(M)$ is isomorphic to one of the following string modules:

$$S_0, M(\beta), M(\eta), \text{ or for } n \geq 1$$

$$\begin{aligned}
A_{1,n} &= M\left(\left(\beta\gamma(\delta^{-1}\eta^{-1}\gamma^{-1}\beta^{-1})^{2^{d-2}-1}\delta^{-1}\eta^{-1}\right)^n \gamma^{-1}\right),\\
A_{2,n} &= M\left(\left(\beta\gamma(\delta^{-1}\eta^{-1}\gamma^{-1}\beta^{-1})^{2^{d-2}-1}\delta^{-1}\eta^{-1}\right)^n\right),\\
A_{3,n} &= M\left(\left(\beta\gamma(\delta^{-1}\eta^{-1}\gamma^{-1}\beta^{-1})^{2^{d-2}-1}\delta^{-1}\eta^{-1}\right)^n \beta\right),\\
A'_{1,n} &= M\left(\left(\eta\delta(\gamma^{-1}\beta^{-1}\delta^{-1}\eta^{-1})^{2^{d-2}-1}\gamma^{-1}\beta^{-1}\right)^n \delta^{-1}\right),\\
A'_{2,n} &= M\left(\left(\eta\delta(\gamma^{-1}\beta^{-1}\delta^{-1}\eta^{-1})^{2^{d-2}-1}\gamma^{-1}\beta^{-1}\right)^n\right),\\
A'_{3,n} &= M\left(\left(\eta\delta(\gamma^{-1}\beta^{-1}\delta^{-1}\eta^{-1})^{2^{d-2}-1}\gamma^{-1}\beta^{-1}\right)^n \eta\right).
\end{aligned}$$

It is straightforward to check that $\underline{\mathrm{End}}_\Lambda(M) = k$ and $\mathrm{Ext}^1_\Lambda(M, M) = 0$ for $M \in \{S_0, M(\beta), M(\eta)\}$. We now demonstrate how to show this for $M = A_{1,n}$ for $n \geq 1$. The other cases are proved similarly. The module $A_{1,n}$ (or more precisely the quiver defining it) is given in Figure 6.2.1. Let $\{x_r\}_{r=0}^{n2^d+1}$ be the corresponding canonical k-basis for $A_{1,n}$ (see §7.1.1). Then there are n sources $h_1, \ldots, h_n$ in the quiver of $A_{1,n}$, corresponding to direct summands of $\mathrm{top}(A_{1,n})$, namely $h_i = x_{(i-1)2^d+2}$ for $1 \leq i \leq n$. There are $n+1$ sinks $f_1, \ldots, f_{n+1}$ in the quiver of $A_{1,n}$, corresponding to direct summands of $\mathrm{soc}(A_{1,n})$, namely $f_j = x_{(j-1)2^d}$ for $1 \leq j \leq n$ and $f_{n+1} = x_{n2^d+1}$. By Remark 6.1.1, each endomorphism of $A_{1,n}$ is a k-linear combination of the elements of

$$\{\mathrm{id}_{A_{1,n}}\} \bigcup \{\lambda_{i,j}, \mu_i, \rho^s_{i,\ell}, \sigma^s_i \mid 1 \leq i, j \leq n, 2 \leq \ell \leq n, 1 \leq s \leq 2^{d-2} - 1\} \tag{6.2.1}$$

where each of these endomorphisms is defined as follows, using Definition 6.1.2:

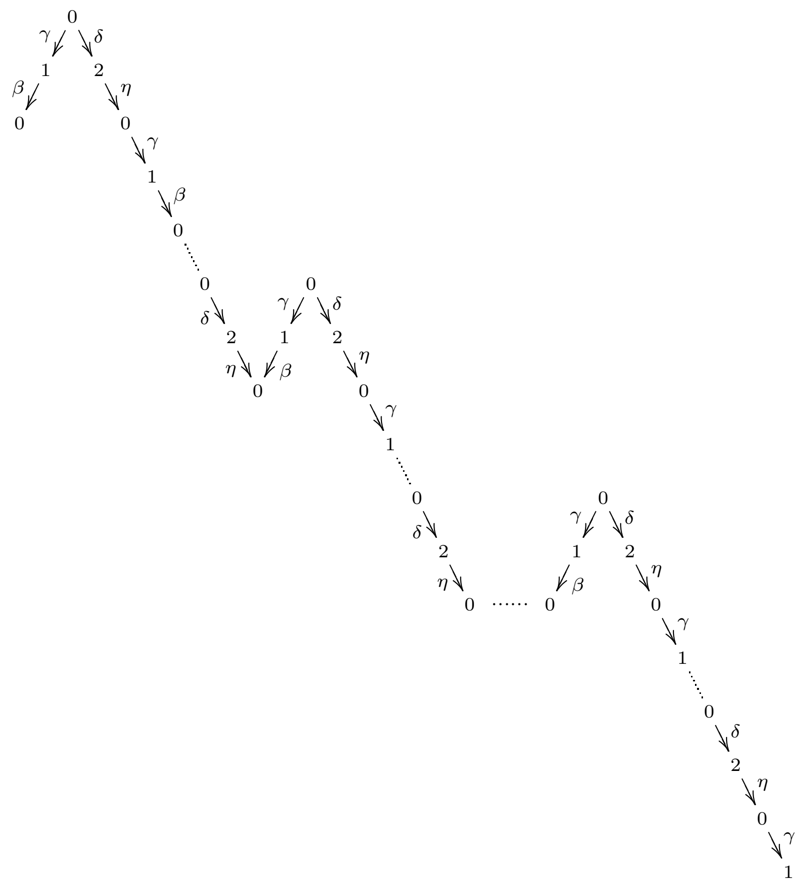
$$\begin{aligned}
\lambda_{i,j} &= \mathrm{end}^{++}_{A_{1,n}}(h_i, f_j, 0);\\
\mu_i &= \mathrm{end}^{-+}_{A_{1,n}}(h_i, f_{n+1}, 1);\\
\rho^s_{i,\ell} &= \mathrm{end}^{++}_{A_{1,n}}(h_i, f_\ell, 4s-2);\\
\sigma^s_i &= \mathrm{end}^{++}_{A_{1,n}}(h_i, f_{n+1}, 4s-1).
\end{aligned} \tag{6.2.2}$$

Let $\alpha_{i,j} = \lambda_{i,j} + \lambda_{i+1,j+1}$ for $1 \leq i, j \leq n-1$. Then it follows that $\alpha_{i,j}$ $(1 \leq i, j \leq n-1)$, $\lambda_{1,j}$ $(2 \leq j \leq n)$, and $\lambda_{n,j}$ $(1 \leq j \leq n)$ each factors through the projective Λ-module P_0. Hence by Lemma 6.1.3, $\lambda_{i,j}$ $(1 \leq i, j \leq n)$ factors through a projective Λ-module. Let $\beta_i = \lambda_{i-1,n} + \mu_i$ for $2 \leq i \leq n$. Then it follows that β_i $(2 \leq i \leq n)$ and μ_1 each factors through P_0. Hence μ_i $(1 \leq i \leq n)$ factors through a projective Λ-module. It also follows that $\rho^s_{i,\ell}$ $(1 \leq i \leq n, 2 \leq \ell \leq n, 1 \leq s \leq 2^{d-2} - 1)$, and σ^s_i $(1 \leq i \leq n, 1 \leq s \leq 2^{d-2} - 1)$ each factors through P_0. Hence $\underline{\mathrm{End}}_\Lambda(A_{1,n}) = k$ for all $n \geq 1$.

Using that $\mathrm{Ext}^1_\Lambda(A_{1,n}, A_{1,n}) = \underline{\mathrm{Hom}}_\Lambda(\Omega(A_{1,n}), A_{1,n})$ and analyzing the quiver defining $\Omega(A_{1,n})$, one shows in a similar fashion that $\mathrm{Ext}^1_\Lambda(A_{1,n}, A_{1,n}) = 0$ for all $n \geq 1$. This completes the proof of Lemma 4.1.4.

Proof of Lemma 4.1.5. Let $i \in \{1, 2\}$ and let $\mathcal{C}_i$ be the component of the stable Auslander-Reiten quiver of Λ containing S_i. We need to show that $\mathcal{C}_i$ is a 3-tube with S_i belonging to its boundary,

FIGURE 6.2.1. The module $A_{1,n}$.



and $\mathcal{C}_i = \Omega(\mathcal{C}_i)$. Moreover, if M belongs to $\mathcal{C}_i$ and has stable endomorphism ring k, we need to show that M is in the Ω-orbit of S_i and $\mathrm{Ext}^1_\Lambda(M,M) = 0$.

We prove this for $i = 1$. (The case $i = 2$ is treated similarly.) The component $\mathfrak{C}_1$ is a 3-tube with boundary consisting of S_1, $\Omega^2(S_1) = P_1/S_1$, and $\Omega^4(S_1) = \Omega(S_1)$. Since Ω maps the boundary to itself, it follows that $\mathfrak{C}_1 = \Omega(\mathfrak{C}_1)$. It is straightforward to check that $\underline{\mathrm{End}}_\Lambda(M) = k$ and $\mathrm{Ext}^1_\Lambda(M,M) = 0$ for M belonging to the boundary of $\mathfrak{C}_1$. It remains to show that all other modules X belonging to $\mathfrak{C}_1$ but not to its boundary have stable endomorphism ring of k-dimension larger than 1.

Using hooks and cohooks (see §7.2), we obtain the following. If X belongs to $\mathfrak{C}_1$ but not to its boundary, then there exist integers j, n with $n \geq 1$ such that $\Omega^j(X)$ is isomorphic to one of the following string modules:

$$X_{1,n} = M\left(\gamma(\delta^{-1}\eta^{-1}\gamma^{-1}\beta^{-1})^{2^{d-2}-1}\delta^{-1}\eta^{-1}\left(\beta\gamma(\delta^{-1}\eta^{-1}\gamma^{-1}\beta^{-1})^{2^{d-2}-1}\delta^{-1}\eta^{-1}\right)^{n-1}\gamma^{-1}\right),$$
$$X_{2,n} = M\left(\gamma(\delta^{-1}\eta^{-1}\gamma^{-1}\beta^{-1})^{2^{d-2}-1}\delta^{-1}\eta^{-1}\left(\beta\gamma(\delta^{-1}\eta^{-1}\gamma^{-1}\beta^{-1})^{2^{d-2}-1}\delta^{-1}\eta^{-1}\right)^{n-1}\right),$$
$$X_{3,n} = M\left(\gamma(\delta^{-1}\eta^{-1}\gamma^{-1}\beta^{-1})^{2^{d-2}-1}\delta^{-1}\eta^{-1}\left(\beta\gamma(\delta^{-1}\eta^{-1}\gamma^{-1}\beta^{-1})^{2^{d-2}-1}\delta^{-1}\eta^{-1}\right)^{n-1}\beta\right).$$

Let $\{x_r^{1,n}\}_{r=0}^{n2^d}$ be the canonical k-basis for $X_{1,n}$. Then $\mathrm{end}_{X_{1,n}}^{-+}(x_1^{1,n}, x_{n2^d}^{1,n}, 1)$ does not factor through any projective Λ-module. Let $\{x_r^{2,n}\}_{r=0}^{n2^d-1}$ (resp. $\{x_r^{3,n}\}_{r=0}^{n2^d}$) be the canonical k-basis for $X_{2,n}$ (resp. $X_{3,n}$). Then, for $\ell \in \{2,3\}$, $\mathrm{end}_{X_{\ell,n}}^{++}(x_1^{\ell,n}, x_{n2^d-1}^{\ell,n}, 0)$ does not factor through any projective Λ-module. This completes the proof of Lemma 4.1.5.

Proof of Lemma 4.1.6. Let $\mathfrak{C}$ be a component of the stable Auslander-Reiten quiver of Λ containing a uniserial module X of length 4. We need to show that $\mathfrak{C}$ and $\Omega(\mathfrak{C})$ are both of type $\mathbb{Z}A_\infty^\infty$, and $\mathfrak{C} = \Omega(\mathfrak{C})$ exactly when $d = 3$. Moreover, if M belongs to $\mathfrak{C} \cup \Omega(\mathfrak{C})$ and has stable endomorphism ring k, we need to show that M is in the Ω-orbit of X and $\mathrm{Ext}^1_\Lambda(M, M) = k$.

There are four uniserial Λ-modules of length 4:

$$X_1 = \begin{matrix} 1 \\ 0 \\ 2 \\ 0 \end{matrix}, \; \Omega^2(X_1) = \begin{matrix} 0 \\ 2 \\ 0 \\ 1 \end{matrix}, \; X_2 = \begin{matrix} 2 \\ 0 \\ 1 \\ 0 \end{matrix}, \; \Omega^2(X_2) = \begin{matrix} 0 \\ 1 \\ 0 \\ 2 \end{matrix}.$$

Let $\mathfrak{C}$ be the component containing X_1. (The case when $\mathfrak{C}$ contains X_2 is treated similarly.) We have seen in Lemma 4.1.3 that X_1 has endomorphism ring k. Moreover, $\Omega(X_1)$ is uniserial of length $2^d - 3$ with descending radical series $(S_1, S_0, S_2, S_0, \ldots, S_1, S_0, S_2, S_0, S_1)$, and thus $\mathrm{Ext}^1_\Lambda(X_1, X_1) = \underline{\mathrm{Hom}}_\Lambda(\Omega(X_1), X_1) = k$.

Using the description of the components of the stable Auslander-Reiten quiver of Λ as in §7.2, we see that $\mathfrak{C}$ is of type $\mathbb{Z}A_\infty^\infty$. Moreover, using hooks and cohooks we can describe the modules M in $\mathfrak{C}$ not belonging to the Ω-orbit of X_1. From this description it follows that $\Omega(X_1)$ lies in $\mathfrak{C}$, and hence $\mathfrak{C} = \Omega(\mathfrak{C})$, if and only if $d = 3$. Similarly to the proof of Lemma 4.1.5 we then construct for each remaining M an endomorphism which does not factor through any projective Λ-module. This completes the proof of Lemma 4.1.6.

6.3. **Components of type $\mathbb{Z}A_\infty^\infty$.** We next consider all components of the stable Auslander-Reiten quiver of type $\mathbb{Z}A_\infty^\infty$. We prove that the only such components containing a module with stable endomorphism ring k are precisely those components containing a module M with $\mathrm{End}_\Lambda(M) = k$ or $\mathrm{End}_\Lambda(\Omega(M)) = k$.

We use the notation of hooks and cohooks from §7.2. Moreover, if S is a string of positive length in a $\mathbb{Z}A_\infty^\infty$-component, we use $S_{h'}$ (resp. ${}_{h'}S$) to denote the string obtained from S by either adding a hook or subtracting a cohook on the right side (resp. left side) of S. We also use $S_{c'}$ (resp. ${}_{c'}S$) to denote the string obtained from S by either adding a cohook or subtracting a hook on the right side (resp. left side) of S. This means that near $M(S)$ the stable Auslander-Reiten component looks as in Figure 6.3.1. Note that if S has minimal length in its stable Auslander-Reiten component then

FIGURE 6.3.1. The stable Auslander-Reiten component near $M(S)$.

$$\begin{tikzcd}
M({}_{c'}S) \arrow[dr] & & M(S_{h'}) \\
& M(S) \arrow[ur] \arrow[dr] & \\
M(S_{c'}) \arrow[ur] & & M({}_{h'}S)
\end{tikzcd}$$

$S_{h'} = S_h$, ${}_{h'}S = {}_hS$, $S_{c'} = S_c$ and ${}_{c'}S = {}_cS$. If none of the projective Λ-modules is uniserial and S has minimal length, then also $S_{h'\cdots h'} = S_{h\cdots h}$, ${}_{h'\cdots h'}S = {}_{h\cdots h}S$, $S_{c'\cdots c'} = S_{c\cdots c}$ and ${}_{c'\cdots c'}S = {}_{c\cdots c}S$. The following Lemma is straightforward.

Lemma 6.3.1. *Let $\Lambda = kQ/I$ be a symmetric special biserial algebra with the following properties:*

a. *the quiver Q contains no double arrows;*
b. *for all $v_1, v_2 \in Q_0$ and $\alpha \in Q_1$ with $s(\alpha) = v_1$ and $e(\alpha) = v_2$, there exists $\tau(\alpha) \in Q_1$ with $s(\tau(\alpha)) = v_2$ and $e(\tau(\alpha)) = v_1$;*
c. *if P is a uniserial projective Λ-module, then P is a string module corresponding to a directed string $\alpha_1\alpha_2\cdots\alpha_\ell$, and*

$$\alpha_1\alpha_2\cdots\alpha_\ell = \tau(\alpha_\ell)\tau(\alpha_{\ell-1})\cdots\tau(\alpha_1);$$

if P is a non-uniserial projective Λ-module, then P corresponds to a relation $p_1 = p_2$ in I for two paths $p_1 = \alpha_1\alpha_2\cdots\alpha_{\ell_1}$ and $p_2 = \beta_1\beta_2\cdots\beta_{\ell_2}$ in kQ, and

$$\{\alpha_1\alpha_2\cdots\alpha_{\ell_1},\ \beta_1\beta_2\cdots\beta_{\ell_2}\} = \{\tau(\alpha_{\ell_1})\tau(\alpha_{\ell_1-1})\cdots\tau(\alpha_1),\ \tau(\beta_{\ell_2})\tau(\beta_{\ell_2-1})\cdots\tau(\beta_1)\}.$$

Let $S = w_1w_2\cdots w_n$ be a string of length $n \geq 1$, and define $\tau(S) = \tau(w_1)^{-1}\tau(w_2)^{-1}\cdots\tau(w_n)^{-1}$. Then $\tau(S)$ is a string. Moreover, suppose $\{x_r\}_{r=0}^n$ (resp. $\{y_r\}_{r=0}^n$) is the canonical k-basis of $M(S)$ (resp. $M(\tau(S))$) relative to the representative S (resp. $\tau(S)$). Let $\epsilon_1, \epsilon_2 \in \{+,-\}$, and let $0 \leq u, v, \ell \leq n$. Define $\rho(+) = -$ and $\rho(-) = +$. If $\mathrm{end}_{M(S)}^{\epsilon_1\,\epsilon_2}(x_u, x_v, \ell)$ is a Λ-endomorphism of $M(S)$, then $\mathrm{end}_{M(\tau(S))}^{\rho(\epsilon_2)\,\rho(\epsilon_1)}(y_v, y_u, \ell)$ is a Λ-endomorphism of $M(\tau(S))$. Moreover, $\mathrm{end}_{M(S)}^{\epsilon_1\,\epsilon_2}(x_u, x_v, \ell)$ factors through a projective Λ-module P if and only if $\mathrm{end}_{M(\tau(S))}^{\rho(\epsilon_2)\,\rho(\epsilon_1)}(y_v, y_u, \ell)$ factors through P.

We are now able to prove the following result.

Proposition 6.3.2. *Let $\Lambda = kQ/I$ where Q and I are as in §3.1. Then the components of the stable Auslander-Reiten quiver of type $\mathbb{Z}A_\infty^\infty$ containing a module with stable endomorphism ring k are precisely the components containing a module M with $\mathrm{End}_\Lambda(M) = k$ or $\mathrm{End}_\Lambda(\Omega(M)) = k$.*

Proof. Let $\mathfrak{C}$ be a component of type $\mathbb{Z}A_\infty^\infty$ of the stable Auslander-Reiten quiver of Λ such that $\mathfrak{C} \cup \Omega(\mathfrak{C})$ contains no simple Λ-module and no uniserial Λ-module of length 4, i.e. by Lemma 4.1.3, $\mathfrak{C} \cup \Omega(\mathfrak{C})$ contains no Λ-module with endomorphism ring k.

Let X be a Λ-module of minimal length in $\mathfrak{C}$. In particular, X (or more precisely the string defining X) cannot start or end in a peak (resp. in a deep). Since P_1 and P_2 are uniserial, this means that X cannot have any of the following forms

(6.3.1)
$$X = \begin{smallmatrix} 1 & & \\ & 0 & \cdots \end{smallmatrix}, \quad X = \begin{smallmatrix} 2 & & \\ & 0 & \cdots \end{smallmatrix}, \quad X = \begin{smallmatrix} & & 1 \\ \cdots & 0 & \end{smallmatrix}, \quad X = \begin{smallmatrix} & & 2 \\ \cdots & 0 & \end{smallmatrix},$$
$$X = \begin{smallmatrix} & 0 & \cdots \\ 1 & & \end{smallmatrix}, \quad X = \begin{smallmatrix} & 0 & \cdots \\ 2 & & \end{smallmatrix}, \quad X = \begin{smallmatrix} \cdots & 0 & \\ & & 1 \end{smallmatrix}, \quad X = \begin{smallmatrix} \cdots & 0 & \\ & & 2 \end{smallmatrix}.$$

Let $\{z_r\}_{r=0}^{\ell(X)}$ be the canonical k-basis of the string module X relative to the chosen representative.

Suppose first X is uniserial. By (6.3.1), $X = M(S)$ where S is one of the following strings for $1 \leq n \leq 2^{d-2} - 1$:

(6.3.2)
$$\begin{aligned} S_{1,1,n} &= \left(\gamma^{-1}\beta^{-1}\delta^{-1}\eta^{-1}\right)^n \gamma^{-1}\beta^{-1}; \\ S_{1,2,n} &= \left(\gamma^{-1}\beta^{-1}\delta^{-1}\eta^{-1}\right)^n; \\ S_{2,2,n} &= \left(\delta^{-1}\eta^{-1}\gamma^{-1}\beta^{-1}\right)^n \delta^{-1}\eta^{-1}; \\ S_{2,1,n} &= \left(\delta^{-1}\eta^{-1}\gamma^{-1}\beta^{-1}\right)^n. \end{aligned}$$

Note that $M(S_{1,1,0})$ (resp. $M(S_{2,2,0})$) lies in the same component of the Auslander-Reiten quiver as a uniserial Λ-module of length 4. If $i \neq j$ in $\{1,2\}$, then $\Omega(M(S_{i,i,2^{d-2}-2}))$ lies in the same Auslander-Reiten component as a uniserial Λ-module of length 4, $M(S_{i,i,2^{d-2}-1})$ lies in the same Auslander-Reiten component as S_j, and $\Omega(M(S_{i,j,2^{d-2}-1}))$ lies in the same Auslander-Reiten component as S_0. On the other hand, if $n < 2^{d-2}-2$, then $\mathrm{end}_{M(S_{i,i,n})}^{++}(z_0, z_{4n+2}, 4n-2)$ does not factor through a projective Λ-module for $i \in \{1,2\}$; and, if $n < 2^{d-2} - 1$, then $\mathrm{end}_{M(S_{i,j,n})}^{++}(z_0, z_{4n}, 4n-4)$

does not factor through a projective Λ-module for $i \neq j$ in $\{1,2\}$. By considering hooks and co-hooks, we see that in both cases, all Λ-modules in the Auslander-Reiten component containing X have stable endomorphism ring of dimension at least 2.

Suppose now X is not uniserial. By Lemma 6.3.1, it is enough to consider X where

$$X = \begin{smallmatrix} 0 & & \\ & 1 & \cdots \end{smallmatrix}, \quad \text{or} \quad X = \begin{smallmatrix} 0 & & \\ & 2 & \cdots \end{smallmatrix}.$$

Let $(\xi_1,\xi_2) = (\eta,\beta)$. If $X = M(S)$ where $S = S_{i,i,n}\xi_i\cdots$ for $i \in \{1,2\}$ and $1 \le n < 2^{d-2}-1$, then $\mathrm{end}_X^{++}(z_0, z_{4n+2}, 4n-2)$ does not factor through any projective Λ-module, and the same is true for $M(S_{h'\cdots h'})$ and $M(S_{c'\cdots c'})$. Similarly, if $X = M(S)$ where $S = S_{i,j,n}\xi_j\cdots$ for $i \neq j$ in $\{1,2\}$ and $1 \le n < 2^{d-2}-1$, then $\mathrm{end}_X^{++}(z_0, z_{4n}, 4n-4)$ does not factor through any projective Λ-module, and the same is true for $M(S_{h'\cdots h'})$ and $M(S_{c'\cdots c'})$. If $X = M(S_{i,i,2^{d-2}-1}\xi_i C_i)$ where $i \in \{1,2\}$ and C_i is some string, then X lies in the same Auslander-Reiten component as $M(C_i)$. Hence X is not of minimal length in $\mathfrak{C}$. The two remaining possibilities for X are

$$X = M(S_{i,i,0}\xi_i\cdots) \text{ for } i \in \{1,2\}, \quad \text{or} \quad X = M(S_{i,j,2^{d-2}-1}\xi_j\cdots) \text{ for } i \neq j \text{ in } \{1,2\}.$$

Let $i \neq j$ in $\{1,2\}$. Then $\Omega(M(S_{i,j,2^{d-2}-1}\xi_j\cdots))$ lies in the same Auslander-Reiten component as a string module of the form $M(\tau(S_{i,i,0}\xi_i\cdots))$, where τ is as in Lemma 6.3.1. Hence, by Lemma 6.3.1, it is enough to consider X of the form $X = M(S_{i,i,0}\xi_i\cdots)$ for $i \in \{1,2\}$.

Let $i = 1$. (The case $i = 2$ is done similarly.) Since X has minimal length in the Auslander-Reiten component $\mathfrak{C}$, this means $X = M(SC)$ where S is one of the following strings:

$$\begin{aligned} S_n^{1,2} &= \gamma^{-1}\beta^{-1}(S_{1,2,n})^{-1}, \quad 1 \le n \le 2^{d-2}-1; \\ S_n^{2,2} &= \gamma^{-1}\beta^{-1}(S_{2,2,n})^{-1}, \quad 0 \le n \le 2^{d-2}-1; \end{aligned} \tag{6.3.3}$$

and C is a string such that the following holds. If C has positive length then $C = \zeta^{-1}\cdots$ for the appropriate arrow ζ in Q, and if $S = S^{2,2}_{2^{d-2}-1}$ then C cannot have length 0.

If $S = S_n^{1,2}$, $1 \le n \le 2^{d-2}-1$, then $\mathrm{end}_X^{++}(z_0,z_2,0)$ does not factor through a projective Λ-module. The same is true for $M((SC)_{c'\cdots c'})$ and for $M((SC)_{h'})$, and, if $n < 2^{d-2}-1$ or C has positive length, also for $M((SC)_{h'\cdots h'})$. If $SC = S = S^{1,2}_{2^{d-2}-1}$, then $(SC)_{h'\cdots h'} = S^{1,2}_{2^{d-2}-1}\eta\delta\gamma^{-1}\beta^{-1}\delta^{-1}\cdots$. Thus $\mathrm{end}^{++}_{M((SC)_{h'\cdots h'})}(z_{2^d},z_2,2)$ does not factor through a projective Λ-module.

If $S = S_n^{2,2}$, $0 \le n \le 2^{d-2}-2$, then $\mathrm{end}_X^{++}(z_0,z_2,0)$ does not factor through a projective Λ-module. The same is also true for $M((SC)_{c'\cdots c'})$ and for $M((SC)_{h'\cdots h'})$.

Now suppose $S = S^{2,2}_{2^{d-2}-1}$. Since X has minimal length in its Auslander-Reiten component, $C = \gamma^{-1}\beta^{-1}C'$ for some string C'. If C' has length 0 or $C' = \delta^{-1}\cdots$, then $\mathrm{end}_X^{++}(z_{2^d},z_2,2)$ does not factor through a projective Λ-module. The same is true for $M((SC)_{c'\cdots c'})$, and, if C' has positive length, also for $M((SC)_{h'\cdots h'})$. If C' has length 0, i.e. $SC = S^{2,2}_{2^{d-2}-1}\gamma^{-1}\beta^{-1}$, then $(SC)_h = (SC)\eta$ and $(SC)_{h'\cdots h'} = (SC)\eta\delta\gamma^{-1}\cdots$. Thus $\mathrm{end}^{++}_{M((SC)_h)}(z_{2^d},z_3,3)$ and $\mathrm{end}^{++}_{M((SC)_{h'\cdots h'})}(z_{2^d},z_4,4)$ do not factor through a projective Λ-module. Now suppose $C' = \eta\cdots$. Then X has the form $X = M(S^{2,2}_{2^{d-2}-1}SC_1)$ where S is one of the strings in (6.3.3) and C_1 has the same properties as the properties of C described below (6.3.3). Hence one continues using similar arguments as above, and thus concludes that the Auslander-Reiten components containing any of these X contain no Λ-module with stable endomorphism ring k. □

6.4. **One-tubes.** Finally, we consider the components of the stable Auslander-Reiten quiver which are 1-tubes. We prove that no 1-tube contains any modules with stable endomorphism ring k. Since for the blocks in question all string modules lie either in components of type $\mathbb{Z}A_\infty^\infty$ or in 3-tubes, all the modules in 1-tubes are band modules. We use the description of band modules from §7.1.

Remark 6.4.1. Let Λ be a special biserial algebra. It follows from [25] that if B is a band, $\lambda \in k-\{0\}$ and $n \ge 2$ is an integer, then $\underline{\mathrm{End}}_\Lambda(M(B,\lambda,n))$ has dimension at least 2. To be more precise, the endomorphisms coming from the circular quiver associated to the band B are parametrized by upper triangular $n \times n$ matrices with equal entries along each diagonal; and these endomorphisms cannot

factor through a projective Λ-module. So if $n \geq 2$, then the k-dimension of the space spanned by these endomorphisms is $n \geq 2$.

Definition 6.4.2. Let $\Lambda = kQ/I$ be a special biserial algebra. Let B be a band for Λ, $\lambda \in k - \{0\}$ and let $M_{B,\lambda} = M(B, \lambda, 1)$. Suppose S is a string such that

i. $B \sim_r ST_1$ with $T_1 = \xi_1^{-1} T_1' \xi_2$, where T_1, T_1' are strings and ξ_1, ξ_2 are arrows in Q; and
ii. $B \sim_r ST_2$ with $T_2 = \zeta_1 T_2' \zeta_2^{-1}$, where T_2, T_2' are strings and ζ_1, ζ_2 are arrows in Q.

Then by [25] there exists an endomorphism of $M_{B,\lambda}$ which factors through $M(S)$. We will call such an endomorphism to be of string type S. Note that there may be several choices of T_1 (resp. T_2) in (i) (resp. (ii)). In other words, there may be more than one endomorphism of string type S. By [25], every endomorphism of $M_{B,\lambda}$ is a k-linear combination of the identity morphism and of endomorphisms of string type S for suitable choices of strings S satisfying (i) and (ii).

Definition 6.4.3. Let $\Lambda = kQ/I$ be a special biserial algebra, and let B be a band for Λ.

i. We call a string C a substring of B if $B \sim_r CC'$ for some string C'.
ii. A substring S of B is called a top-socle piece of B if
   a. $S = \alpha_\ell^{-1} \cdots \alpha_2^{-1} \alpha_1^{-1}$ for $\ell \geq 1$ and arrows $\alpha_1, \alpha_2, \ldots, \alpha_\ell$ in Q, and
   b. $B \sim_r ST$ for some string T where $T = \xi T' \zeta$ and ξ, ζ are arrows in Q.

   Note that $B \sim_r C_0 C_1^{-1} C_2 C_3^{-1} \cdots C_s^{-1}$ where $s \geq 1$ is odd and $C_0, C_1, \ldots, C_s$ are top-socle pieces of B.
iii. If S is a top-socle piece of B and E is a simple Λ-module which is isomorphic to the top (resp. socle) of $M(S)$, we also say that S has top (resp. socle) isomorphic to E.

In the proof of the following Proposition, we will often use the equality sign instead of the more precise $\sim_r$.

Proposition 6.4.4. *Let $\Lambda = kQ/I$ where Q and I are as in §3.1. Then each module M which lies in a component of the stable Auslander-Reiten quiver which is a 1-tube satisfies* $\dim_k \underline{\mathrm{End}}_\Lambda(M) \geq 2$.

Proof. Let B be a band for Λ such that possibly $\underline{\mathrm{End}}_\Lambda(M(B, \lambda, n)) = k$. By Remark 6.4.1, we only need to consider $M_{B,\lambda} = M(B, \lambda, 1)$ for $\lambda \in k - \{0\}$.

It follows from the shape of the quiver Q that the only top-socle pieces that can occur in B must have top and socle isomorphic to S_0. This means they have the form, using the notation from (6.3.2):

$$S_{i,i,n} \text{ for } 0 \leq n \leq 2^{d-2} - 1, \quad \text{and } S_{i,j,n} \text{ for } 1 \leq n \leq 2^{d-2} - 1, \quad \text{where } i \neq j \text{ in } \{1, 2\}.$$

Let now $i \neq j$ in $\{1, 2\}$. If $S_{i,i,n}$ for $0 < n < 2^{d-2} - 1$ (resp. $S_{i,j,n}$ for $1 < n < 2^{d-2} - 1$) is a top-socle piece of B, then $M_{B,\lambda}$ has an endomorphism of string type $S_{i,i,n-1}$ (resp. of string type $S_{i,j,n-1}$) which does not factor through a projective module. This implies that the only top-socle pieces that can occur in B are (again using the notation from (6.3.2))

$$S_{i,i,0}, S_{i,i,2^{d-2}-1}, \quad \text{and } S_{i,j,1}, S_{i,j,2^{d-2}-1} \quad \text{where } i \neq j \text{ in } \{1, 2\}. \tag{6.4.1}$$

Let $i \neq j$ in $\{1, 2\}$, and suppose $S_{i,j,1}$ (resp. $S_{i,j,2^{d-2}-1}$) is a top-socle piece of B. If $S_{j,i,1}$ is also a top-socle piece of B, then $M_{B,\lambda}$ has an endomorphism of string type $S_{j,j,0}$ which does not factor through a projective module. On the other hand, if $S_{j,i,2^{d-2}-1}$ is also a top-socle piece of B, then $M_{B,\lambda}$ has an endomorphism of string type $S_{j,j,0}$ (resp. $S_{j,j,2^{d-2}-2}$) which does not factor through a projective module. This means that if $S_{i,j,1}$ (resp. $S_{i,j,2^{d-2}-1}$) is a top-socle piece of B, then neither $S_{j,i,1}$ nor $S_{j,i,2^{d-2}-1}$ can be a top-socle piece of B.

We claim that this makes it impossible for either $S_{i,j,1}$ or $S_{i,j,2^{d-2}-1}$ to occur as top-socle piece of B. We demonstrate this for the case $(i,j) = (1,2)$. Let $\ell \in \{1, 2^{d-2} - 1\}$, and suppose $S_{i,j,\ell}$ is a top-socle piece of B. This means $B = S_{1,2,\ell} C$ for some string $C = C_1^{-1} C_2 C_3^{-1} \cdots C_s^{-1}$ where $s \geq 1$ is odd and $C_1, C_2, \ldots, C_s$ are top-socle pieces of B. But then C_u must be of the form $S_{1,1,1}$ or $S_{1,1,2^{d-2}-1}$ for odd u, and of the form $S_{2,2,1}$ or $S_{2,2,2^{d-2}-1}$ for even u. In particular, C_s is of the form $S_{1,1,1}$ or $S_{1,1,2^{d-2}-1}$. But this contradicts that B is a band, since $C_s^{-1} S_{1,2,\ell}$ is not a valid word.

Hence, by (6.4.1), the only top-socle pieces that can occur in B are

(6.4.2) $$S_{i,i,0}, S_{i,i,2^{d-2}-1} \quad \text{for } i \in \{1,2\}.$$

Let $i \neq j$ in $\{1,2\}$, and suppose $S_{i,i,0}$ is a top-socle piece of B. Then $B = S_{i,i,0}C$ for some string $C = C_1^{-1}C_2C_3^{-1}\cdots C_s^{-1}$ where $s \geq 1$ is odd and $C_1, C_2, \ldots, C_s$ are top-socle pieces of B.

We claim that C_1 and C_s must both be $S_{j,j,2^{d-2}-1}$. This can be shown as follows. Suppose first $C_s = S_{j,j,0}$. Then it follows that $C_{s-1} = S_{i,i,2^{d-2}-1}$ and $C_1 = S_{j,j,2^{d-2}-1}$, since otherwise $M_{B,\lambda}$ has an endomorphism of string type 1_0 (i.e. the string of length 0 corresponding to the vertex 0) which does not factor through a projective module. But then, to ensure that B is a band, one of two things must be true: Either there is an odd u_0 with $C_{u_0}^{-1}C_{u_0+1} = S_{j,j,2^{d-2}-1}^{-1}S_{i,i,2^{d-2}-1}$, in which case $M_{B,\lambda}$ has an endomorphism of string type $S_{j,j,0}^{-1}S_{i,i,0}$ which does not factor through a projective module. Or there is an odd u_0 with $C_{u_0-1}C_{u_0}^{-1} = S_{i,i,0}S_{j,j,0}^{-1}$, in which case $C_{u_0+1} = S_{i,i,2^{d-2}-1}$ which means $M_{B,\lambda}$ has an endomorphism of string type $S_{i,i,0}S_{j,j,0}^{-1}S_{i,i,0}$ which does not factor through a projective module. Hence we get a contradiction, which means $C_s = S_{j,j,2^{d-2}-1}$. The same argument shows that if $u < s$ is odd then at least one of C_u and C_{u+1} must be in $\{S_{1,1,2^{d-2}-1}, S_{2,2,2^{d-2}-1}\}$. Suppose now that $C_1 = S_{j,j,0}$. Then we must have $C_2 = S_{i,i,2^{d-2}-1}$. Moreover, there must exist an odd u_0 such that $C_{u_0-1}C_{u_0}^{-1} = S_{i,i,2^{d-2}-1}S_{j,j,2^{d-2}-1}^{-1}$. But this means that $M_{B,\lambda}$ has an endomorphism of string type $S_{i,i,0}S_{j,j,0}^{-1}$ which does not factor through a projective module. Thus C_1 and C_s are both $S_{j,j,2^{d-2}-1}$.

Since B cannot be the power of a smaller word, we see that if $s > 1$ then $M_{B,\lambda}$ has an endomorphism of string type $S_{i,i,0}S_{j,j,2^{d-2}-1}^{-1}S_{i,i,0}$ which does not factor through a projective module. On the other hand, if $s = 1$ then $B = S_{i,i,0}S_{j,j,2^{d-2}-1}^{-1}$, and $M_{B,\lambda}$ has an endomorphism of string type 1_0 (i.e. the string of length 0 corresponding to the vertex 0) which does not factor through a projective module.

Hence neither $S_{1,1,0}$ nor $S_{2,2,0}$ is a top-socle piece of B. Thus the only possible band is $B = S_{1,1,2^{d-2}-1}S_{2,2,2^{d-2}-1}^{-1}$. But then $M_{B,\lambda}$ has an endomorphism of string type $S_{1,2,2^{d-2}-1}$ which does not factor through a projective module. So in all cases, $\underline{\mathrm{End}}_\Lambda(M_{B,\lambda})$ has k-dimension at least 2, which completes the proof of Proposition 6.4.4. □

7. Background: Special biserial algebras

In this section, we give a short introduction to special biserial algebras. For more background material, we refer to [12]. Let k be an algebraically closed field of characteristic $p > 0$, let Q be a finite quiver and let I be an admissible ideal in the path algebra kQ.

Definition 7.1. A finite dimensional basic k-algebra $\Lambda = kQ/I$ is called special biserial if the following conditions are satisfied:

i. Any vertex of Q is starting point (resp. end point) of at most two arrows.
ii. For a given arrow β in Q, there is at most one arrow γ with $\beta\gamma \notin I$, and there is at most one arrow α with $\alpha\beta \notin I$.

If additionally I is generated by paths, Λ is called a string algebra.

If Λ is a special biserial algebra and $\mathcal{P}$ is a full set of representatives of the projective indecomposable Λ-modules which are also injective and not uniserial, then $\bar{\Lambda} = \Lambda/(\oplus_{P\in\mathcal{P}}\mathrm{soc}(P))$ is a string algebra. Furthermore, the indecomposable $\bar{\Lambda}$-modules are exactly the indecomposable Λ-modules which are not isomorphic to any $P \in \mathcal{P}$.

7.1. **Indecomposable modules for string algebras.** Let $\Lambda = kQ/I$ be a basic string algebra. Then all indecomposable Λ-modules are either string or band modules (see e.g. [12, §3]). The definitions are as follows.

Given an arrow β in Q with starting point $s(\beta)$ and end point $e(\beta)$, denote by β^{-1} a formal inverse of β. In particular, $s(\beta^{-1}) = e(\beta)$, $e(\beta^{-1}) = s(\beta)$, and $(\beta^{-1})^{-1} = \beta$. A word w is a sequence $w_1 \cdots w_n$, where w_i is either an arrow or a formal inverse such that $s(w_i) = e(w_{i+1})$ for

$1 \le i \le n-1$. Define $s(w) = s(w_n)$, $e(w) = e(w_1)$ and $w^{-1} = w_n^{-1} \cdots w_1^{-1}$. For each vertex u in Q there exists an empty word 1_u of length 0 with $e(1_u) = s(1_u) = u$ and $(1_u)^{-1} = 1_u$. Denote the set of all words by $\mathcal{W}$, and the set of all non-empty words w with $e(w) = s(w)$ by $\mathcal{W}_r$. In the following, Greek letters inside words always denote arrows.

7.1.1. *Strings and string modules.* Let $\sim_s$ be the equivalence relation on $\mathcal{W}$ with $w \sim_s w'$ if and only if $w = w'$ or $w^{-1} = w'$. Then strings are representatives $w \in \mathcal{W}$ of the equivalence classes under $\sim_s$ with the following property: Either $w = 1_u$ or $w = w_1 \cdots w_n$, where $w_i \neq w_{i+1}^{-1}$ for $1 \le i \le n-1$ and no subword of w or its formal inverse belongs to I.

Let $C = w_1 \cdots w_n$ be a string of length n and let Q_C be the linear quiver

$$Q_C = \;\cdot \overset{w_1}{\text{—}} \cdot \;\cdots\; \cdot \overset{w_n}{\text{—}} \cdot$$

where $\cdot \overset{w_i}{\text{—}} \cdot \;=\; \cdot \xleftarrow{\beta} \cdot$ if $w_i = \beta$ is an arrow, and $\cdot \overset{w_i}{\text{—}} \cdot \;=\; \cdot \xrightarrow{\beta} \cdot$ if $w_i = \beta^{-1}$ is a formal inverse. Then the representation of Q_C which assigns to each vertex the vector space k and to each arrow the identity map defines an indecomposable Λ-module, called the string module $M(C)$ corresponding to the string C. More precisely, there is a k-basis $\{z_0, z_1, \ldots, z_n\}$ of $M(C)$ such that the action of Λ on $M(C)$ is given by the following representation $\varphi_C : \Lambda \to \mathrm{Mat}(n+1, k)$. Let $v(i) = e(w_{i+1})$ for $0 \le i \le n-1$ and $v(n) = s(w_n)$. Then for each vertex u and for each arrow α in Q

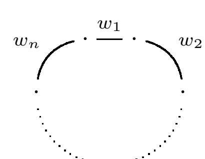
$$\varphi_C(u)(z_i) = \left\{ \begin{array}{ccl} z_i & , & \text{if } v(i) = u \\ 0 & , & \text{else} \end{array} \right\} \text{ and } \varphi_C(\alpha)(z_i) = \left\{ \begin{array}{ccl} z_{i-1} & , & \text{if } w_i = \alpha \\ z_{i+1} & , & \text{if } w_{i+1} = \alpha^{-1} \\ 0 & , & \text{else} \end{array} \right\}.$$

We will call φ_C the canonical representation and $\{z_0, z_1, \ldots, z_n\}$ the canonical k-basis for $M(C)$ relative to the representative C. Note that $M(C) \cong M(C^{-1})$.

The string modules $M(1_u)$, u a vertex of Q, correspond bijectively to the isomorphism classes of the simple Λ-modules. We say a string $C = w_1 \cdots w_n$ is directed if all w_i are arrows. For each vertex u of Q, there exist at most two directed strings of maximal length starting in u. Let these be C_1 and C_2. Then the projective indecomposable Λ-module $P(u)$ is the string module $M(C_1 C_2^{-1})$. Dually, the injective indecomposable module $E(u)$ is the string module $M(D_1^{-1} D_2)$ where D_1 and D_2 are the directed strings of maximal length ending in u.

7.1.2. *Bands and band modules.* Let $w = w_1 \cdots w_n \in \mathcal{W}_r$. Then, for $0 \le i \le n-1$, the i-th rotation of w is defined to be the word $\rho_i(w) = w_{i+1} \cdots w_n w_1 \cdots w_i$. Let $\sim_r$ be the equivalence relation on $\mathcal{W}_r$ such that $w \sim_r w'$ if and only if $w = \rho_i(w')$ for some i or $w^{-1} = \rho_j(w')$ for some j. Then bands are representatives $w \in \mathcal{W}_r$ of the equivalence classes under $\sim_r$ with the following property: $w = w_1 \cdots w_n$, $n \ge 1$, with $w_i \neq w_{i+1}^{-1}$ and $w_n \neq w_1^{-1}$, such that w is not a power of a smaller word, and, for all positive integers m, no subword of w^m or its formal inverse belongs to I.

Let $B = w_1 \cdots w_n$ be a band of length n. We may assume that w_1 is an arrow, by rotating and possibly inverting. Let Q_B^c be the circular quiver

$Q_B^c =$ 

where $\cdot \overset{w_i}{\text{—}} \cdot \;=\; \cdot \overset{\beta}{\curvearrowleft} \cdot$ points counter-clockwise if $w_i = \beta$ is an arrow, and $\cdot \overset{w_i}{\text{—}} \cdot \;=\; \cdot \overset{\beta}{\curvearrowright} \cdot$ points clockwise if $w_i = \beta^{-1}$ is a formal inverse. Let $m > 0$ be an integer and $\lambda \in k^*$. Then the representation of Q_B^c which assigns to each vertex the vector space k^m, to w_1 the indecomposable Jordan matrix $J_m(\lambda)$, and to w_i, $2 \le i \le n$, the identity map defines an indecomposable Λ-module, called the band module $M(B, \lambda, m)$ corresponding to B, λ and m. Note that for all i, j

$$M(B, \lambda, m) \cong M(\rho_i(B), \lambda, m) \cong M(\rho_j(B)^{-1}, \lambda, m).$$

7.2. **Auslander-Reiten components.** Let $\Lambda = kQ/I$ be a basic string algebra. Then in each component of the Auslander-Reiten quiver of Λ there are either only string modules or only band modules. The band modules all lie in 1-tubes. The string modules can lie in periodic components or in non-periodic components. We now describe the irreducible morphisms between string modules using hooks and cohooks. Let S be a string.

We say that S starts on a peak provided there is no arrow β with $S\beta$ a string, and that S starts in a deep provided there is no arrow γ with $S\gamma^{-1}$ a string. Dually, we say that S ends on a peak provided there is no arrow β with $\beta^{-1}S$ a string, and that S ends in a deep provided there is no arrow γ with γS a string.

If S does not start on a peak, there is a unique arrow β and a unique maximal directed string M such that $S_h = S\beta M^{-1}$ starts in a deep. We say S_h is obtained from S by adding a hook on the right side. Dually, if S does not end on a peak, there is a unique arrow β and a unique directed string M such that ${}_hS = M\beta^{-1}S$ ends in a deep. We say ${}_hS$ is obtained from S by adding a hook on the left side.

If S does not start in a deep, there is a unique arrow γ and a unique maximal directed string N such that $S_c = S\gamma^{-1}N$ starts on a peak. We say S_c is obtained from S by adding a cohook on the right side. Dually, if S does not end in a deep, there is a unique arrow γ and a unique directed string N such that ${}_cS = N^{-1}\gamma S$ ends on a peak. We say ${}_cS$ is obtained from S by adding a cohook on the left side.

All irreducible morphisms between string modules are either canonical injections

$$M(S) \to M(S_h), \quad \text{or} \quad M(S) \to M({}_hS),$$

or canonical projections

$$M(S_c) \to M(S), \quad \text{or} \quad M({}_cS) \to M(S).$$

If Λ is a basic special biserial algebra which is self-injective, then $\Lambda/\mathrm{soc}(\Lambda)$ is a string algebra. Moreover, the stable Auslander-Reiten quiver of Λ is equal to the Auslander-Reiten quiver of $\Lambda/\mathrm{soc}(\Lambda)$. In case Λ is Morita equivalent to a block with dihedral defect groups, the periodic components containing string modules are either 1-tubes or 3-tubes, and the non-periodic components all have type $\mathbb{Z}A_\infty^\infty$.

References

[1] J. L. Alperin, Local representation theory. Modular representations as an introduction to the local representation theory of finite groups. Cambridge Studies in Advanced Mathematics, vol. 11, Cambridge University Press, Cambridge, 1986.

[2] M. Auslander, I. Reiten and S. Smalø, Representation theory of Artin algebras. Cambridge Studies in Advanced Mathematics, vol. 36, Cambridge University Press, Cambridge, 1995.

[3] D. J. Benson, Representations and cohomology. I: Basic representation theory of finite groups and associative algebras. Cambridge Studies in Advanced Mathematics, vol. 30, Cambridge University Press, Cambridge, 1991.

[4] F. M. Bleher, Universal deformation rings and Klein four defect groups. Trans. Amer. Math. Soc. 354 (2002), 3893–3906.

[5] F. M. Bleher, Deformations and derived equivalences. Proc. Amer. Math. Soc. 134 (2006), 2503–2510.

[6] F. M. Bleher and T. Chinburg, Universal deformation rings and cyclic blocks. Math. Ann. 318 (2000), 805–836.

[7] F. M. Bleher and T. Chinburg, Universal deformation rings need not be complete intersections. C. R. Math. Acad. Sci. Paris 342 (2006), 229–232.

[8] F. M. Bleher and T. Chinburg, Universal deformation rings need not be complete intersections. Math. Ann. 337 (2007), 739–767.

[9] R. Brauer, On 2-blocks with dihedral defect groups. Symposia Mathematica, vol. XIII (Convegno di Gruppi e loro Rappresentazioni, INDAM, Rome, 1972), pp. 367–393, Academic Press, London, 1974.

[10] C. Breuil, B. Conrad, F. Diamond and R. Taylor, On the modularity of elliptic curves over $\mathbb{Q}$: Wild 3-adic exercises. J. Amer. Math. Soc. 14 (2001), 843–939.

[11] M. Broué, Equivalences of blocks of group algebras. In: Finite dimensional algebras and related topics (Ottawa, ON, 1992), NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., vol. 424, Kluwer Acad. Publ., Dordrecht, 1994, pp. 1–26.

[12] M. C. R. Butler and C. M. Ringel, Auslander-Reiten sequences with few middle terms and applications to string algebras. Comm. Algebra 15 (1987), 145–179.

[13] J. Byszewski, A universal deformation ring which is not a complete intersection ring. C. R. Math. Acad. Sci. Paris 343 (2006), 565–568.

[14] T. Chinburg, Can deformation rings of group representations not be local complete intersections? In: Problems from the Workshop on Automorphisms of Curves. Edited by Gunther Cornelissen and Frans Oort, with contributions by I. Bouw, T. Chinburg, G. Cornelissen, C. Gasbarri, D. Glass, C. Lehr, M. Matignon, F. Oort, R. Pries and S. Wewers. Rend. Sem. Mat. Univ. Padova 113 (2005), 129–177.

[15] S. B. Conlon, Certain representation algebras. J. Austral. Math. Soc. 5 (1965), 83–99.

[16] G. Cornell, J. H. Silverman and G. Stevens (eds.), Modular Forms and Fermat's Last Theorem (Boston, 1995). Springer-Verlag, Berlin-Heidelberg-New York, 1997.

[17] C. W. Curtis and I. Reiner, Methods of representation theory. Vol. I. With applications to finite groups and orders. John Wiley & Sons, Inc., New York, 1981.

[18] B. de Smit and H. W. Lenstra, Explicit construction of universal deformation rings. In: Modular Forms and Fermat's Last Theorem (Boston, 1995), Springer-Verlag, Berlin-Heidelberg-New York, 1997, pp. 313–326.

[19] M. Dettweiler and S. Wewers, Variation of local systems and parabolic cohomology. Israel J. Math. 156 (2006), 157–185.

[20] K. Erdmann, Blocks of Tame Representation Type and Related Algebras. Lecture Notes in Mathematics, vol. 1428, Springer-Verlag, Berlin-Heidelberg-New York, 1990.

[21] P. Fong, A note on splitting fields of representations of finite groups. Illinois J. Math. 7 (1963) 515–520.

[22] D. Gorenstein and J. H. Walter, The characterization of finite groups with dihedral Sylow 2-subgroups. I, II, III. J. Algebra 2 (1965) 85–151, 218–270, 354–393.

[23] B. Huppert, Endliche Gruppen. I. Die Grundlehren der Mathematischen Wissenschaften, Band 134, Springer-Verlag, Berlin-New York, 1967.

[24] S. König and A. Zimmermann, Derived equivalences for group rings. With contributions by Bernhard Keller, Markus Linckelmann, Jeremy Rickard and Raphaël Rouquier. Lecture Notes in Mathematics, vol. 1685. Springer-Verlag, Berlin, 1998.

[25] H. Krause, Maps between tree and band modules. J. Algebra 137 (1991), 186–194.

[26] M. Linckelmann, A derived equivalence for blocks with dihedral defect groups. J. Algebra 164 (1994), 244–255.

[27] B. Mazur, Deforming Galois representations. In: Galois groups over $\mathbb{Q}$ (Berkeley, 1987), Springer-Verlag, Berlin-Heidelberg-New York, 1989, pp. 385–437.

[28] J. Rickard, Derived equivalences as derived functors. J. London Math. Soc. 43 (1991), 37–48.

[29] J. Rickard, Splendid equivalences: derived categories and permutation modules. Proc. London Math. Soc. 72 (1996), 331–358.

[30] J. P. Serre, Linear representations of finite groups. Translated from the second French edition by Leonard L. Scott. Graduate Texts in Mathematics, vol. 42, Springer-Verlag, New York-Heidelberg, 1977.

[31] T. Shiina, Regular Galois realizations of $\mathrm{PSL}_2(p^2)$ over $\mathbb{Q}(T)$. In: Galois theory and modular forms (Tokyo, 2001), Dev. Math., 11, Kluwer Acad. Publ., Boston, 2004, pp. 125–142.

[32] R. Taylor and A. Wiles, Ring-theoretic properties of certain Hecke algebras. Ann. of Math. 141 (1995), 553–572.

[33] G. Wiese, On projective linear groups over finite fields as Galois groups over the rational numbers. ArXiv math.NT/0606732.

[34] A. Wiles, Modular elliptic curves and Fermat's last theorem. Ann. of Math. 141 (1995), 443–551.

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